Chapter 3

Special Techniques

Problem 3.1

The argument is exactly the same as in Sect. 3.1.4, except that since $z < R$, $\sqrt{z^2 + R^2 - 2zR} = (R - z)$, instead of $(z - R)$. Hence $V_{\text{ave}} = \frac{q}{4\pi\epsilon_0}\frac{1}{2zR}[(z + R) - (R - z)] = \boxed{\frac{1}{4\pi\epsilon_0}\frac{q}{R}.}$ If there is more than one charge inside the sphere, the average potential due to interior charges is $\frac{1}{4\pi\epsilon_0}\frac{Q_{\text{enc}}}{R}$, and the average due to exterior charges is V_{center}, so $V_{\text{ave}} = V_{\text{center}} + \frac{Q_{\text{enc}}}{4\pi\epsilon_0 R}$. ✓

Problem 3.2

A stable equilibrium is a point of local minimum in the potential energy. Here the potential energy is qV. But we know that Laplace's equation allows no local minima for V. What *looks* like a minimum, in the figure, must in fact be a saddle point, and the box "leaks" through the center of each face.

Problem 3.3

Laplace's equation in *spherical* coordinates, for V dependent only on r, reads:

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = 0 \Rightarrow r^2\frac{dV}{dr} = c \text{ (constant)} \Rightarrow \frac{dV}{dr} = \frac{c}{r^2} \Rightarrow \boxed{V = -\frac{c}{r} + k.}$$

Example: potential of a uniformly charged sphere.

In *cylindrical* coordinates: $\nabla^2 V = \frac{1}{s}\frac{d}{ds}\left(s\frac{dV}{ds}\right) = 0 \Rightarrow s\frac{dV}{ds} = c \Rightarrow \frac{dV}{ds} = \frac{c}{s} \quad \Rightarrow \quad \boxed{V = c\ln s + k.}$

Example: potential of a long wire.

Problem 3.4

Same as proof of second uniqueness theorem, up to the equation $\oint_S V_3\mathbf{E}_3 \cdot d\mathbf{a} = -\int_{\mathcal{V}}(E_3)^2\,d\tau$. But on each surface, either $V_3 = 0$ (if V is specified on the surface), or else $E_{3_\perp} = 0$ (if $\frac{\partial V}{\partial n} = -E_\perp$ is specified). So $\int_{\mathcal{V}}(E_3)^2 = 0$, and hence $\mathbf{E}_2 = \mathbf{E}_1$. qed

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Problem 3.5

Putting $U = T = V_3$ into Green's identity:

$\int_{\mathcal{V}}[V_3\nabla^2 V_3 + \nabla V_3 \cdot \nabla V_3]\,d\tau = \oint_S V_3\nabla V_3 \cdot d\mathbf{a}$. But $\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = -\frac{\rho}{\epsilon_0} + \frac{\rho}{\epsilon_0} = 0$, and $\nabla V_3 = -\mathbf{E}_3$. So $\int_{\mathcal{V}} E_3^2\,d\tau = -\oint_S V_2\mathbf{E}_3 \cdot d\mathbf{a}$, and the rest is the same as before.

Problem 3.6

Place image charges $+2q$ at $z = -d$ and $-q$ at $z = -3d$. Total force on $+q$ is

$$\mathbf{F} = \frac{q}{4\pi\epsilon_0}\left[\frac{-2q}{(2d)^2} + \frac{2q}{(4d)^2} + \frac{-q}{(6d)^2}\right]\hat{\mathbf{z}} = \frac{q^2}{4\pi\epsilon_0 d^2}\left(-\frac{1}{2} + \frac{1}{8} - \frac{1}{36}\right)\hat{\mathbf{z}} = \boxed{-\frac{1}{4\pi\epsilon_0}\left(\frac{29q^2}{72d^2}\right)\hat{\mathbf{z}}.}$$

Problem 3.7

(a) From Fig. 3.13: $\mathscr{r} = \sqrt{r^2 + a^2 - 2ra\cos\theta}$; $\mathscr{r}' = \sqrt{r^2 + b^2 - 2rb\cos\theta}$. Therefore:

$$\begin{aligned}\frac{q'}{\mathscr{r}'} &= -\frac{R}{a}\frac{q}{\sqrt{r^2 + b^2 - 2rb\cos\theta}} \quad \text{(Eq. 3.15), while } b = \frac{R^2}{a} \text{ (Eq. 3.16).}\\ &= -\frac{q}{\left(\frac{a}{R}\right)\sqrt{r^2 + \frac{R^4}{a^2} - 2r\frac{R^2}{a}\cos\theta}} = -\frac{q}{\sqrt{\left(\frac{ar}{R}\right)^2 + R^2 - 2ra\cos\theta}}.\end{aligned}$$

Therefore:

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}} + \frac{q'}{\mathscr{r}'}\right) = \boxed{\frac{q}{4\pi\epsilon_0}\left\{\frac{1}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} - \frac{1}{\sqrt{R^2 + (ra/R)^2 - 2ra\cos\theta}}\right\}.}$$

Clearly, when $r = R$, $V \to 0$.

(b) $\sigma = -\epsilon_0\frac{\partial V}{\partial n}$ (Eq. 2.49). In this case, $\frac{\partial V}{\partial n} = \frac{\partial V}{\partial r}$ at the point $r = R$. Therefore,

$$\begin{aligned}\sigma(\theta) &= -\epsilon_0\left(\frac{q}{4\pi\epsilon_0}\right)\left\{-\frac{1}{2}(r^2 + a^2 - 2ra\cos\theta)^{-3/2}(2r - 2a\cos\theta)\right.\\ &\qquad \left.\left. + \frac{1}{2}\left(R^2 + (ra/R)^2 - 2ra\cos\theta\right)^{-3/2}\left(\frac{a^2}{R^2}2r - 2a\cos\theta\right)\right\}\right|_{r=R}\\ &= -\frac{q}{4\pi}\left\{-(R^2 + a^2 - 2Ra\cos\theta)^{-3/2}(R - a\cos\theta) + \left(R^2 + a^2 - 2Ra\cos\theta\right)^{-3/2}\left(\frac{a^2}{R} - a\cos\theta\right)\right\}\\ &= \frac{q}{4\pi}(R^2 + a^2 - 2Ra\cos\theta)^{-3/2}\left[R - a\cos\theta - \frac{a^2}{R} + a\cos\theta\right]\\ &= \boxed{\frac{q}{4\pi R}(R^2 - a^2)(R^2 + a^2 - 2Ra\cos\theta)^{-3/2}.}\\ q_{\text{induced}} &= \int\sigma\,da = \frac{q}{4\pi R}(R^2 - a^2)\int(R^2 + a^2 - 2Ra\cos\theta)^{-3/2}R^2\sin\theta\,d\theta\,d\phi\\ &= \frac{q}{4\pi R}(R^2 - a^2)2\pi R^2\left.\left[-\frac{1}{Ra}(R^2 + a^2 - 2Ra\cos\theta)^{-1/2}\right]\right|_0^\pi\\ &= \frac{q}{2a}(a^2 - R^2)\left[\frac{1}{\sqrt{R^2 + a^2 + 2Ra}} - \frac{1}{\sqrt{R^2 + a^2 - 2Ra}}\right].\end{aligned}$$

But $a > R$ (else q would be *inside*), so $\sqrt{R^2 + a^2 - 2Ra} = a - R$.

$$\begin{aligned}&= \frac{q}{2a}(a^2 - R^2)\left[\frac{1}{(a+R)} - \frac{1}{(a-R)}\right] = \frac{q}{2a}\left[(a - R) - (a + R)\right] = \frac{q}{2a}(-2R)\\ &= \boxed{-\frac{qR}{a} = q'.}\end{aligned}$$

(c) The force on q, due to the sphere, is the same as the force of the image charge q', to wit:

$$F = \frac{1}{4\pi\epsilon_0}\frac{qq'}{(a-b)^2} = \frac{1}{4\pi\epsilon_0}\left(-\frac{R}{a}q^2\right)\frac{1}{(a-R^2/a)^2} = -\frac{1}{4\pi\epsilon_0}\frac{q^2Ra}{(a^2-R^2)^2}.$$

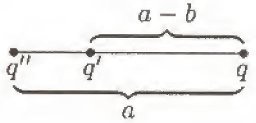
To bring q in from infinity to a, then, we do work

$$W = \frac{q^2R}{4\pi\epsilon_0}\int_{\infty}^{a}\frac{\bar{a}}{(\bar{a}^2-R^2)^2}\,d\bar{a} = \frac{q^2R}{4\pi\epsilon_0}\left[-\frac{1}{2}\frac{1}{(\bar{a}^2-R^2)}\right]\Bigg|_{\infty}^{a} = \boxed{-\frac{1}{4\pi\epsilon_0}\frac{q^2R}{2(a^2-R^2)}.}$$

Problem 3.8

Place a second image charge, q'', at the *center* of the sphere; this will not alter the fact that the sphere is an *equipotential*, but merely *increase* that potential from zero to $V_0 = \frac{1}{4\pi\epsilon_0}\frac{q''}{R}$;

$\boxed{q'' = 4\pi\epsilon_0 V_0 R \text{ at center of sphere.}}$



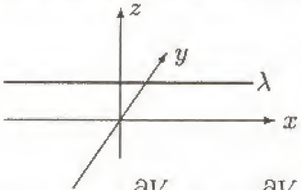
For a *neutral* sphere, $q' + q'' = 0$.

$$\begin{aligned}
F &= \frac{1}{4\pi\epsilon_0}q\left(\frac{q''}{a^2}+\frac{q'}{(a-b)^2}\right) = \frac{qq'}{4\pi\epsilon_0}\left(-\frac{1}{a^2}+\frac{1}{(a-b)^2}\right)\\
&= \frac{qq'}{4\pi\epsilon_0}\frac{b(2a-b)}{a^2(a-b)^2} = \frac{q(-Rq/a)}{4\pi\epsilon_0}\frac{(R^2/a)(2a-R^2/a)}{a^2(a-R^2/a)^2}\\
&= -\boxed{\frac{q^2}{4\pi\epsilon_0}\left(\frac{R}{a}\right)^3\frac{(2a^2-R^2)}{(a^2-R^2)^2}.}
\end{aligned}$$

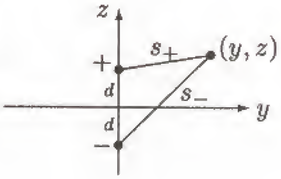
(Drop the minus sign, because the problem asks for the force of *attraction*.)

Problem 3.9

(a) Image problem: λ above, $-\lambda$ below. Potential was found in Prob. 2.47:



$$V(y,z) = \frac{2\lambda}{4\pi\epsilon_0}\ln(s_-/s_+) = \frac{\lambda}{4\pi\epsilon_0}\ln(s_-^2/s_+^2)$$
$$= \boxed{\frac{\lambda}{4\pi\epsilon_0}\ln\left\{\frac{y^2+(z+d)^2}{y^2+(z-d)^2}\right\}}$$



(b) $\sigma = -\epsilon_0\frac{\partial V}{\partial n}$. Here $\frac{\partial V}{\partial n} = \frac{\partial V}{\partial z}$, evaluated at $z = 0$.

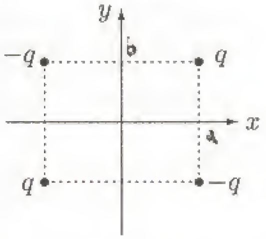
$$\begin{aligned}
\sigma(y) &= -\epsilon_0\frac{\lambda}{4\pi\epsilon_0}\left\{\frac{1}{y^2+(z+d)^2}2(z+d) - \frac{1}{y^2+(z-d)^2}2(z-d)\right\}\Bigg|_{z=0}\\
&= -\frac{2\lambda}{4\pi}\left\{\frac{d}{y^2+d^2} - \frac{-d}{y^2+d^2}\right\} = \boxed{-\frac{\lambda d}{\pi(y^2+d^2)}.}
\end{aligned}$$

Check: Total charge induced on a strip of width l parallel to the y axis:

$$\begin{aligned}
q_{\text{ind}} &= -\frac{l\lambda d}{\pi}\int_{-\infty}^{\infty}\frac{1}{y^2+d^2}\,dy = -\frac{l\lambda d}{\pi}\left[\frac{1}{d}\tan^{-1}\left(\frac{y}{d}\right)\right]\Bigg|_{-\infty}^{\infty} = -\frac{l\lambda d}{\pi}\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right]\\
&= -\lambda l. \quad \text{Therefore } \lambda_{\text{ind}} = -\lambda, \text{ as it should be.}
\end{aligned}$$

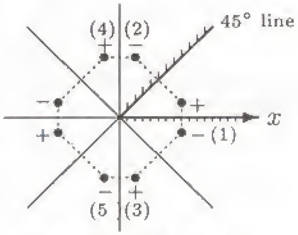
Problem 3.10

The image configuration is as shown.



$$\boxed{\begin{aligned} V(x,y) &= \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{\sqrt{(x-a)^2+(y-b)^2+z^2}}+\frac{1}{\sqrt{(x+a)^2+(y+b)^2+z^2}}\right. \\ &\quad \left. -\frac{1}{\sqrt{(x+a)^2+(y-b)^2+z^2}}-\frac{1}{\sqrt{(x-a)^2+(y+b)^2+z^2}}\right\}. \end{aligned}}$$

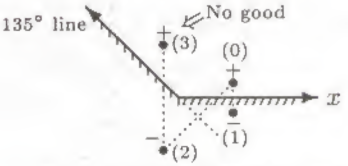
For this to work, $\boxed{\theta \text{ must be and integer divisor of } 180^\circ.}$ Thus 180°, 90°, 60°, 45°, etc., are OK, but no others. It works for 45°, say, with the charges as shown.



why it works for $\theta = 45°$

(Note the strategy: to make the x axis an equipotential ($V = 0$), you place the image charge (1) in the reflection point. To make the 45° line an equipotential, you place charge (2) at the image point. But that screws up the x axis, so you must now insert image (3) to balance (2). Moreover, to make the 45° line $V = 0$ you also need (4), to balance (1). But now, to restore the x axis to $V = 0$ you need (5) to balance (4), and so on.

The reason this doesn't work for *arbitrary* angles is that you are eventually forced to place an image charge *within the original region of interest*, and that's not allowed—*all* images must go *outside* the region, or you're no longer dealing with the same problem at all.)



why it *doesn't* work for $\theta = 135°$

Problem 3.11

From Prob. 2.47 (with $y_0 \to d$): $\boxed{V = \frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{(x+a)^2+y^2}{(x-a)^2+y^2}\right]}$, where $a^2 = {y_0}^2 - R^2 \Rightarrow$ $\boxed{a = \sqrt{d^2-R^2},}$

and

$$\left\{\begin{array}{l} a\coth(2\pi\epsilon_0 V_0/\lambda) = d \\ a\,\mathrm{csch}(2\pi\epsilon_0 V_0/\lambda) = R \end{array}\right\} \Rightarrow \text{(dividing)} \quad \frac{d}{R} = \cosh\left(\frac{2\pi\epsilon_0 V_0}{\lambda}\right), \text{ or } \quad \boxed{\lambda = \frac{2\pi\epsilon_0 V_0}{\cosh^{-1}(d/R)}.}$$

Problem 3.12

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a}\sin(n\pi y/a) \quad \text{(Eq. 3.30)}, \quad \text{where} \quad C_n = \frac{2}{a}\int_0^a V_0(y)\sin(n\pi y/a)\,dy \quad \text{(Eq. 3.34)}.$$

In this case $V_0(y) = \left\{\begin{array}{ll} +V_0, & \text{for } 0 < y < a/2 \\ -V_0, & \text{for } a/2 < y < a \end{array}\right\}$. Therefore,

$$\begin{aligned} C_n &= \frac{2}{a}V_0\left\{\int_0^{a/2}\sin(n\pi y/a)\,dy - \int_{a/2}^{a}\sin(n\pi y/a)\,dy\right\} = \frac{2V_0}{a}\left\{-\left.\frac{\cos(n\pi y/a)}{(n\pi/a)}\right|_0^{a/2} + \left.\frac{\cos(n\pi y/a)}{(n\pi/a)}\right|_{a/2}^{a}\right\} \\ &= \frac{2V_0}{n\pi}\left\{-\cos\left(\frac{n\pi}{2}\right) + \cos(0) + \cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right\} = \frac{2V_0}{n\pi}\left\{1 + (-1)^n - 2\cos\left(\frac{n\pi}{2}\right)\right\}. \end{aligned}$$

The term in curly brackets is:

$$\left\{\begin{array}{lcl} n=1 & : & 1-1-2\cos(\pi/2)=0, \\ n=2 & : & 1+1-2\cos(\pi)=4, \\ n=3 & : & 1-1-2\cos(3\pi/2)=0, \\ n=4 & : & 1+1-2\cos(2\pi)=0, \end{array}\right\} \text{etc. (Zero if } n \text{ is odd or divisible by 4, otherwise 4.)}$$

Therefore

$$C_n = \begin{cases} 8V_0/n\pi, & n = 2, 6, 10, 14, \text{etc. (in general, } 4j+2, \text{ for } j = 0, 1, 2, \ldots), \\ 0, & \text{otherwise.} \end{cases}$$

So

$$\boxed{V(x,y) = \frac{8V_0}{\pi} \sum_{n=2,6,10,\ldots} \frac{e^{-n\pi x/a}\sin(n\pi y/a)}{n}} = \frac{8V_0}{\pi} \sum_{j=0}^{\infty} \frac{e^{-(4j+2)\pi x/a}\sin[(4j+2)\pi y/a]}{(4j+2)}.$$

Problem 3.13

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,5,\ldots} \frac{1}{n} e^{-n\pi x/a}\sin(n\pi y/a) \quad \text{(Eq. 3.36);} \quad \sigma = -\epsilon_0 \frac{\partial V}{\partial n} \quad \text{(Eq. 2.49).}$$

So

$$\begin{aligned} \sigma(y) &= -\epsilon_0 \frac{\partial}{\partial x}\left\{\frac{4V_0}{\pi}\sum \frac{1}{n} e^{-n\pi x/a}\sin(n\pi y/a)\right\}\bigg|_{x=0} = -\epsilon_0 \frac{4V_0}{\pi} \sum \frac{1}{n}\left(-\frac{n\pi}{a}\right) e^{-n\pi x/a}\sin(n\pi y/a)\bigg|_{x=0} \\ &= \boxed{\frac{4\epsilon_0 V_0}{a} \sum_{n=1,3,5,\ldots} \sin(n\pi y/a).} \end{aligned}$$

Or, using the closed form 3.37:

$$\begin{aligned} V(x,y) &= \frac{2V_0}{\pi}\tan^{-1}\left(\frac{\sin(\pi y/a)}{\sinh(\pi x/a)}\right) \Rightarrow \sigma = -\epsilon_0 \frac{2V_0}{\pi} \frac{1}{1+\frac{\sin^2(\pi y/a)}{\sinh^2(\pi x/a)}} \left(\frac{-\sin(\pi y/a)}{\sinh^2(\pi x/a)}\right) \frac{\pi}{a}\cosh(\pi x/a)\bigg|_{x=0} \\ &= \frac{2\epsilon_0 V_0}{a} \frac{\sin(\pi y/a)\cosh(\pi x/a)}{\sin^2(\pi y/a)+\sinh^2(\pi x/a)}\bigg|_{x=0} = \boxed{\frac{2\epsilon_0 V_0}{a}\frac{1}{\sin(\pi y/a)}.} \end{aligned}$$

Summation of series Eq. 3.36

$$V(x,y) = \frac{4V_0}{\pi} I, \text{ where } I \equiv \sum_{n=1,3,5,\ldots} \frac{1}{n} e^{-n\pi x/a}\sin(n\pi y/a).$$

Now $\sin w = \mathcal{I}m\left(e^{iw}\right)$, so

$$I = \mathcal{I}m \sum \frac{1}{n} e^{-n\pi x/a} e^{in\pi y/a} = \mathcal{I}m \sum \frac{1}{n}\mathcal{Z}^n,$$

where $\mathcal{Z} \equiv e^{-\pi(x-iy)/a}$. Now

$$\begin{aligned} \sum_{1,3,5,\ldots} \frac{1}{n}\mathcal{Z}^n &= \sum_{j=0}^{\infty} \frac{1}{(2j+1)} \mathcal{Z}^{(2j+1)} = \int_0^{\mathcal{Z}} \left\{\sum_{j=0}^{\infty} u^{2j}\right\} du \\ &= \int_0^{\mathcal{Z}} \frac{1}{1-u^2}\, du = \frac{1}{2}\ln\left(\frac{1+\mathcal{Z}}{1-\mathcal{Z}}\right) = \frac{1}{2}\ln\left(Re^{i\theta}\right) = \frac{1}{2}\left(\ln R + i\theta\right), \end{aligned}$$

where $Re^{i\theta} = \frac{1+\mathcal{Z}}{1-\mathcal{Z}}$. Therefore

$$\begin{aligned} I &= \mathcal{I}m\left\{\frac{1}{2}(\ln R + i\,\theta)\right\} = \frac{1}{2}\theta. \quad \text{But } \frac{1+\mathcal{Z}}{1-\mathcal{Z}} = \frac{1+e^{-\pi(x-iy)/a}}{1-e^{-\pi(x-iy)/a}} = \frac{\left(1+e^{-\pi(x-iy)/a}\right)\left(1-e^{-\pi(x+iy)/a}\right)}{\left(1-e^{-\pi(x-iy)/a}\right)\left(1-e^{-\pi(x+iy)/a}\right)} \\ &= \frac{1+e^{-\pi x/a}\left(e^{i\pi y/a}-e^{-i\pi y/a}\right)-e^{-2\pi x/a}}{\left|1-e^{-\pi(x-iy)/a}\right|^2} = \frac{1+2ie^{-\pi x/a}\sin(\pi y/a)-e^{-2\pi x/a}}{\left|1-e^{-\pi(x-iy)/a}\right|^2}, \end{aligned}$$

so

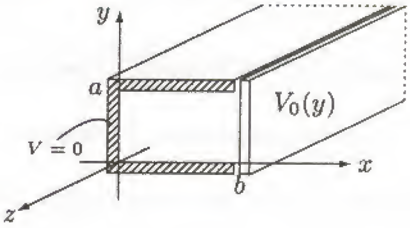
$$\tan\theta = \frac{2e^{-\pi x/a}\sin(\pi y/a)}{1-e^{-2\pi x/a}} = \frac{2\sin(\pi y/a)}{e^{\pi x/a}-e^{-\pi x/a}} = \frac{\sin(\pi y/a)}{\sinh(\pi x/a)}.$$

Therefore

$$I = \frac{1}{2}\tan^{-1}\left(\frac{\sin(\pi y/a)}{\sinh(\pi x/a)}\right), \text{ and } \boxed{V(x,y) = \frac{2V_0}{\pi}\tan^{-1}\left(\frac{\sin(\pi y/a)}{\sinh(\pi x/a)}\right).}$$

Problem 3.14

(a) $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = 0$, with boundary conditions



$$\left\{\begin{array}{ll} \text{(i)} & V(x,0) = 0, \\ \text{(ii)} & V(x,a) = 0, \\ \text{(iii)} & V(0,y) = 0, \\ \text{(iv)} & V(b,y) = V_0(y). \end{array}\right\}$$

As in Ex. 3.4, separation of variables yields

$$V(x,y) = \left(Ae^{kx} + Be^{-kx}\right)(C\sin ky + D\cos ky).$$

Here (i)⇒ $D = 0$, (iii)⇒ $B = -A$, (ii)⇒ ka is an integer multiple of π:

$$V(x,y) = AC\left(e^{n\pi x/a} - e^{-n\pi x/a}\right)\sin(n\pi y/a) = (2AC)\sinh(n\pi x/a)\sin(n\pi y/a).$$

But $(2AC)$ is a constant, and the most general linear combination of separable solutions consistent with (i), (ii), (iii) is

$$\boxed{V(x,y) = \sum_{n=1}^{\infty} C_n \sinh(n\pi x/a)\sin(n\pi y/a).}$$

It remains to determine the coefficients C_n so as to fit boundary condition (iv):

$$\sum C_n \sinh(n\pi b/a)\sin(n\pi y/a) = V_0(y). \text{ Fourier's trick} \Rightarrow C_n \sinh(n\pi b/a) = \frac{2}{a}\int_0^a V_0(y)\sin(n\pi y/a)\,dy.$$

Therefore

$$\boxed{C_n = \frac{2}{a\sinh(n\pi b/a)}\int_0^a V_0(y)\sin(n\pi y/a)\,dy.}$$

(b) $C_n = \dfrac{2}{a\sinh(n\pi b/a)}V_0\displaystyle\int_0^a \sin(n\pi y/a)\,dy = \dfrac{2V_0}{a\sinh(n\pi b/a)} \times \left\{\begin{array}{ll} 0, & \text{if } n \text{ is even,} \\ \frac{2a}{n\pi}, & \text{if } n \text{ is odd.}\end{array}\right\}$

$$\boxed{V(x,y) = \frac{4V_0}{\pi}\sum_{n=1,3,5,\dots}\frac{\sinh(n\pi x/a)\sin(n\pi y/a)}{n\sinh(n\pi b/a)}.}$$

Problem 3.15

Same format as Ex. 3.5, only the boundary conditions are:

$$\left\{\begin{array}{llll} \text{(i)} & V=0 & \text{when} & x=0, \\ \text{(ii)} & V=0 & \text{when} & x=a, \\ \text{(iii)} & V=0 & \text{when} & y=0, \\ \text{(iv)} & V=0 & \text{when} & y=a, \\ \text{(v)} & V=0 & \text{when} & z=0, \\ \text{(vi)} & V=V_0 & \text{when} & z=a. \end{array}\right\}$$

This time we want sinusoidal functions in x and y, exponential in z:

$$X(x) = A\sin(kx) + B\cos(kx), \quad Y(y) = C\sin(ly) + D\cos(ly), \quad Z(z) = Ee^{\sqrt{k^2+l^2}z} + Ge^{-\sqrt{k^2+l^2}z}.$$

(i)$\Rightarrow B = 0$; (ii)$\Rightarrow k = n\pi/a$; (iii)$\Rightarrow D = 0$; (iv)$\Rightarrow l = m\pi/a$; (v)$\Rightarrow E + G = 0$. Therefore

$$Z(z) = 2E\sinh(\pi\sqrt{n^2+m^2}z/a).$$

Putting this all together, and combining the constants, we have:

$$V(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}\sin(n\pi x/a)\sin(m\pi y/a)\sinh(\pi\sqrt{n^2+m^2}z/a).$$

It remains to evaluate the constants $C_{n,m}$, by imposing boundary condition (vi):

$$V_0 = \sum\sum\left[C_{n,m}\sinh(\pi\sqrt{n^2+m^2})\right]\sin(n\pi x/a)\sin(m\pi y/a).$$

According to Eqs. 3.50 and 3.51:

$$C_{n,m}\sinh\left(\pi\sqrt{n^2+m^2}\right) = \left(\frac{2}{a}\right)^2 V_0\int_0^a\int_0^a \sin(n\pi x/a)\sin(m\pi y/a)\,dx\,dy = \left\{\begin{array}{ll} 0, & \text{if } n \text{ or } m \text{ is even,} \\ \dfrac{16V_0}{\pi^2 nm}, & \text{if both are odd.}\end{array}\right\}$$

Therefore

$$\boxed{V(x,y,z) = \frac{16V_0}{\pi^2}\sum_{n=1,3,5,\dots}\sum_{m=1,3,5,\dots}\frac{1}{nm}\sin(n\pi x/a)\sin(m\pi y/a)\frac{\sinh\left(\pi\sqrt{n^2+m^2}z/a\right)}{\sinh\left(\pi\sqrt{n^2+m^2}\right)}.}$$

Problem 3.16

$$\begin{aligned}
P_3(x) &= \frac{1}{8\cdot 6}\frac{d^3}{dx^3}\left(x^2-1\right)^3 = \frac{1}{48}\frac{d^2}{dx^2}3\left(x^2-1\right)^2 2x = \frac{1}{8}\frac{d^2}{dx^2}x\left(x^2-1\right)^2 \\
&= \frac{1}{8}\frac{d}{dx}\left[\left(x^2-1\right)^2 + 2x\left(x^2-1\right)2x\right] = \frac{1}{8}\frac{d}{dx}\left[\left(x^2-1\right)\left(x^2-1+4x^2\right)\right] \\
&= \frac{1}{8}\frac{d}{dx}\left[\left(x^2-1\right)\left(5x^2-1\right)\right] = \frac{1}{8}\left[2x\left(5x^2-1\right)+\left(x^2-1\right)10x\right] \\
&= \frac{1}{4}\left(5x^3-x+5x^3-5x\right) = \frac{1}{4}\left(10x^3-6x\right) = \boxed{\frac{5}{2}x^3-\frac{3}{2}x.}
\end{aligned}$$

We need to show that $P_3(\cos\theta)$ satisfies

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) = -l(l+1)P, \text{ with } l=3,$$

where $P_3(\cos\theta) = \frac{1}{2}\cos\theta\left(5\cos^2\theta-3\right)$.

$$\begin{aligned}
\frac{dP_3}{d\theta} &= \frac{1}{2}\left[-\sin\theta\left(5\cos^2\theta-3\right)+\cos\theta(10\cos\theta(-\sin\theta)\right] = -\frac{1}{2}\sin\theta\left(5\cos^2\theta-3+10\cos^2\theta\right) \\
&= -\frac{3}{2}\sin\theta\left(5\cos^2\theta-1\right).
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial\theta}\left(\sin\theta\frac{dP_3}{d\theta}\right) &= -\frac{3}{2}\frac{d}{d\theta}\left[\sin^2\theta\left(5\cos^2\theta-1\right)\right] = -\frac{3}{2}\left[2\sin\theta\cos\theta\left(5\cos^2\theta-1\right)+\sin^2\theta\left(-10\cos\theta\sin\theta\right)\right] \\
&= -3\sin\theta\cos\theta\left[5\cos^2\theta-1-5\sin^2\theta\right].
\end{aligned}$$

$$\begin{aligned}
\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) &= -3\cos\theta\left[5\cos^2-1-5\left(1-\cos^2\theta\right)\right] = -3\cos\theta\left(10\cos^2\theta-6\right) \\
&= -3\cdot 4\cdot\frac{1}{2}\cos\theta\left(5\cos^2\theta-3\right) = -l(l+1)P_3. \quad \text{qed}
\end{aligned}$$

$$\int_{-1}^{1} P_1(x)P_3(x)\,dx = \int_{-1}^{1}(x)\frac{1}{2}\left(5x^3-3x\right)\,dx = \frac{1}{2}\left.\left(x^5-x^3\right)\right|_{-1}^{1} = \frac{1}{2}(1-1+1-1) = 0. \checkmark$$

Problem 3.17

(a) *Inside:* $V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$ (Eq. 3.66) where

$$A_l = \frac{(2l+1)}{2R^l}\int_0^{\pi} V_0(\theta)P_l(\cos\theta)\sin\theta\,d\theta \quad \text{(Eq. 3.69)}.$$

In this case $V_0(\theta) = V_0$ comes outside the integral, so

$$A_l = \frac{(2l+1)V_0}{2R^l}\int_0^{\pi} P_l(\cos\theta)\sin\theta\,d\theta.$$

But $P_0(\cos\theta) = 1$, so the integral can be written

$$\int_0^\pi P_0(\cos\theta)P_l(\cos\theta)\sin\theta\,d\theta = \left\{\begin{array}{ll} 0, & \text{if } l \neq 0 \\ 2, & \text{if } l = 0 \end{array}\right\} \quad \text{(Eq. 3.68).}$$

Therefore

$$A_l = \left\{\begin{array}{ll} 0, & \text{if } l \neq 0 \\ V_0, & \text{if } l = 0 \end{array}\right\}.$$

Plugging this into the general form:

$$V(r,\theta) = A_0\,r^0 P_0(\cos\theta) = \boxed{V_0.}$$

The potential is *constant throughout the sphere.*

Outside: $V(r,\theta) = \sum\limits_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$ (Eq. 3.72), where

$$\begin{aligned} B_l &= \frac{(2l+1)}{2} R^{l+1} \int_0^\pi V_0(\theta) P_l(\cos\theta)\sin\theta\,d\theta \quad \text{(Eq. 3.73).} \\ &= \frac{(2l+1)}{2} R^{l+1} V_0 \int_0^\pi P_l(\cos\theta)\sin\theta\,d\theta = \left\{\begin{array}{ll} 0, & \text{if } l \neq 0 \\ RV_0, & \text{if } l = 0 \end{array}\right\}. \end{aligned}$$

Therefore $\boxed{V(r,\theta) = V_0\frac{R}{r}}$ (i.e. equals V_0 at $r = R$, then falls off like $\frac{1}{r}$).

(b)

$$V(r,\theta) = \left\{\begin{array}{lll} \sum\limits_{l=0}^{\infty} A_l r^l P_l(\cos\theta), & \text{for } r \leq R & \text{(Eq. 3.78)} \\ \sum\limits_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta), & \text{for } r \geq R & \text{(Eq. 3.79)} \end{array}\right\},$$

where

$$B_l = R^{2l+1} A_l \quad \text{(Eq. 3.81)}$$

and

$$\begin{aligned} A_l &= \frac{1}{2\epsilon_0 R^{l-1}} \int_0^\pi \sigma_0(\theta) P_l(\cos\theta)\sin\theta\,d\theta \quad \text{(Eq. 3.84)} \\ &= \frac{1}{2\epsilon_0 R^{l-1}} \sigma_0 \int_0^\pi P_l(\cos\theta)\sin\theta\,d\theta = \left\{\begin{array}{ll} 0, & \text{if } l \neq 0 \\ R\sigma_0/\epsilon_0, & \text{if } l = 0 \end{array}\right\}. \end{aligned}$$

Therefore

$$\boxed{V(r,\theta) = \left\{\begin{array}{ll} \dfrac{R\sigma_0}{\epsilon_0}, & \text{for } r \leq R \\ \dfrac{R^2\sigma_0}{\epsilon_0}\dfrac{1}{r}, & \text{for } r \geq R \end{array}\right\}.}$$

Note: in terms of the total charge $Q = 4\pi R^2 \sigma_0$,

$$V(r,\theta) = \left\{ \begin{array}{ll} \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{R}, & \text{for } r \leq R \\ \\ \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{r}, & \text{for } r \geq R \end{array} \right\}$$

Problem 3.18

$$V_0(\theta) = k\cos(3\theta) = k\left[4\cos^3\theta - 3\cos\theta\right] = k\left[\alpha P_3(\cos\theta) + \beta P_1(\cos\theta)\right].$$

(I know that any 3rd order polynomial can be expressed as a linear combination of the first four Legendre polynomials; in this case, since the polynomial is *odd*, I only need P_1 and P_3.)

$$4\cos^3\theta - 3\cos\theta = \alpha\left[\frac{1}{2}\left(5\cos^3\theta - 3\cos\theta\right)\right] + \beta\cos\theta = \frac{5\alpha}{2}\cos^3\theta + \left(\beta - \frac{3}{2}\alpha\right)\cos\theta,$$

so

$$4 = \frac{5\alpha}{2} \Rightarrow \alpha = \frac{8}{5}; \quad -3 = \beta - \frac{3}{2}\alpha = \beta - \frac{3}{2}\cdot\frac{8}{5} = \beta - \frac{12}{5} \Rightarrow \beta = \frac{12}{5} - 3 = -\frac{3}{5}.$$

Therefore

$$V_0(\theta) = \frac{k}{5}\left[8P_3(\cos\theta) - 3P_1(\cos\theta)\right].$$

Now

$$V(r,\theta) = \left\{ \begin{array}{lll} \displaystyle\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta), & \text{for } r \leq R & \text{(Eq. 3.66)} \\ \displaystyle\sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta), & \text{for } r \geq R & \text{(Eq. 3.71)} \end{array} \right\},$$

where

$$\begin{aligned} A_l &= \frac{(2l+1)}{2R^l}\int_0^{\pi} V_0(\theta)P_l(\cos\theta)\sin\theta\,d\theta \quad \text{(Eq. 3.69)} \\ &= \frac{(2l+1)}{2R^l}\frac{k}{5}\left\{8\int_0^{\pi} P_3(\cos\theta)P_l(\cos\theta)\sin\theta\,d\theta - 3\int_0^{\pi} P_1(\cos\theta)P_l(\cos\theta)\sin\theta\,d\theta\right\} \\ &= \frac{k}{5}\frac{(2l+1)}{2R^l}\left\{8\frac{2}{(2l+1)}\,\delta_{l3} - 3\frac{2}{(2l+1)}\,\delta_{l1}\right\} = \frac{k}{5}\frac{1}{R^l}\left[8\,\delta_{l3} - 3\,\delta_{l1}\right] \\ &= \left\{ \begin{array}{ll} 8k/5R^3, & \text{if } l = 3 \\ -3k/5R, & \text{if } l = 1 \end{array} \right\} \text{(zero otherwise).} \end{aligned}$$

Therefore

$$V(r,\theta) = -\frac{3k}{5R}rP_1(\cos\theta) + \frac{8k}{5R^3}r^3P_3(\cos\theta) = \boxed{\frac{k}{5}\left[8\left(\frac{r}{R}\right)^3 P_3(\cos\theta) - 3\left(\frac{r}{R}\right)P_1(\cos\theta)\right]},$$

or

$$\frac{k}{5}\left\{8\left(\frac{r}{R}\right)^3\frac{1}{2}\left[5\cos^3\theta - 3\cos\theta\right] - 3\left(\frac{r}{R}\right)\cos\theta\right\} \Rightarrow \boxed{V(r,\theta) = \frac{k}{5}\frac{r}{R}\cos\theta\left\{4\left(\frac{r}{R}\right)^2\left[5\cos^2\theta - 3\right] - 3\right\}}$$

(for $r \le R$). Meanwhile, $B_l = A_l R^{2l+1}$ (Eq. 3.81—this follows from the continuity of V at R). Therefore

$$B_l = \left\{ \begin{array}{ll} 8kR^4/5, & \text{if } l = 3 \\ -3kR^2/5, & \text{if } l = 1 \end{array} \right\} \quad \text{(zero otherwise).}$$

So

$$V(r,\theta) = \frac{-3kR^2}{5}\frac{1}{r^2}P_1(\cos\theta) + \frac{8kR^4}{5}\frac{1}{r^4}P_3(\cos\theta) = \boxed{\frac{k}{5}\left[8\left(\frac{R}{r}\right)^4 P_3(\cos\theta) - 3\left(\frac{R}{r}\right)^2 P_1(\cos\theta)\right]},$$

or

$$\boxed{V(r,\theta) = \frac{k}{5}\left(\frac{R}{r}\right)^2 \cos\theta \left\{4\left(\frac{R}{r}\right)^2 [5\cos^2\theta - 3] - 3\right\}}$$

(for $r \ge R$). Finally, using Eq. 3.83:

$$\begin{aligned}
\sigma(\theta) &= \epsilon_0 \sum_{l=0}^{\infty}(2l+1)A_l R^{l-1} P_l(\cos\theta) = \epsilon_0\left[3A_1P_1 + 7A_3R^2P_3\right] \\
&= \epsilon_0\left[3\left(-\frac{3k}{5R}\right)P_1 + 7\left(\frac{8k}{5R^3}\right)R^2P_3\right] = \boxed{\frac{\epsilon_0 k}{5R}[-9P_1(\cos\theta) + 56P_3(\cos\theta)]} \\
&= \frac{\epsilon_0 k}{5R}\left[-9\cos\theta + \frac{56}{2}(5\cos^3\theta - 3\cos\theta)\right] = \frac{\epsilon_0 k}{5R}\cos\theta[-9 + 28\cdot 5\cos^2\theta - 28\cdot 3] \\
&= \boxed{\frac{\epsilon_0 k}{5R}\cos\theta\left[140\cos^2\theta - 93\right].}
\end{aligned}$$

Problem 3.19

Use Eq. 3.83: $\sigma(\theta) = \epsilon_0 \sum_{l=0}^{\infty}(2l+1)A_l R^{l-1}P_l(\cos\theta)$. But Eq. 3.69 says: $A_l = \frac{2l+1}{2R^l}\int_0^\pi V_0(\theta)P_l(\cos\theta)\sin\theta\, d\theta$. Putting them together:

$$\sigma(\theta) = \frac{\epsilon_0}{2R}\sum_{l=0}^{\infty}(2l+1)^2 C_l P_l(\cos\theta), \quad \text{with } C_l = \int_0^\pi V_0(\theta)P_l(\cos\theta)\sin\theta\, d\theta. \quad \text{qed}$$

Problem 3.20

Set $V = 0$ on the equatorial plane, far from the sphere. Then the potential is the same as Ex. 3.8 ***plus*** the potential of a uniformly charged spherical shell:

$$\boxed{V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta + \frac{1}{4\pi\epsilon_0}\frac{Q}{r}.}$$

Problem 3.21

(a) $V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$ $(r > R)$, so $V(r,0) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(1) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} = \frac{\sigma}{2\epsilon_0}\left[\sqrt{r^2+R^2} - r\right]$.

Since $r > R$ in this region, $\sqrt{r^2+R^2} = r\sqrt{1+(R/r)^2} = r\left[1 + \frac{1}{2}(R/r)^2 - \frac{1}{8}(R/r)^4 + \dots\right]$, so

$$\sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} = \frac{\sigma}{2\epsilon_0} r\left[1 + \frac{1}{2}\frac{R^2}{r^2} - \frac{1}{8}\frac{R^4}{r^4} + \dots - 1\right] = \frac{\sigma}{2\epsilon_0}\left(\frac{R^2}{2r} - \frac{R^4}{8r^3} + \dots\right).$$

Comparing like powers of r, I see that $B_0 = \frac{\sigma R^2}{4\epsilon_0}$, $B_1 = 0$, $B_2 = -\frac{\sigma R^4}{16\epsilon_0}, \dots$. Therefore

$$\boxed{\begin{aligned} V(r,\theta) &= \frac{\sigma R^2}{4\epsilon_0}\left[\frac{1}{r} - \frac{R^2}{4r^3}P_2(\cos\theta) + \dots\right], \\ &= \frac{\sigma R^2}{4\epsilon_0 r}\left[1 - \frac{1}{8}\left(\frac{R}{r}\right)^2 (3\cos^2\theta - 1) + \dots\right], \end{aligned}}$$

(for $r > R$).

(b) $V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$ $(r < R)$. In the northern hemispere, $0 \le \theta \le \pi/2$,

$$V(r,0) = \sum_{l=0}^{\infty} A_l r^l = \frac{\sigma}{2\epsilon_0}\left[\sqrt{r^2+R^2} - r\right].$$

Since $r < R$ in this region, $\sqrt{r^2+R^2} = R\sqrt{1+(r/R)^2} = R\left[1 + \frac{1}{2}(r/R)^2 - \frac{1}{8}(r/R)^4 + \dots\right]$. Therefore

$$\sum_{l=0}^{\infty} A_l r^l = \frac{\sigma}{2\epsilon_0}\left[R + \frac{1}{2}\frac{r^2}{R} - \frac{1}{8}\frac{r^4}{R^3} + \dots - r\right].$$

Comparing like powers: $A_0 = \frac{\sigma}{2\epsilon_0}R$, $A_1 = -\frac{\sigma}{2\epsilon_0}$, $A_2 = \frac{\sigma}{2\epsilon_0 R}, \dots$, so

$$\boxed{\begin{aligned} V(r,\theta) &= \frac{\sigma}{2\epsilon_0}\left[R - rP_1(\cos\theta) + \frac{1}{2R}P_2(\cos\theta) + \dots\right], \\ &= \frac{\sigma R}{2\epsilon_0}\left[1 - \left(\frac{r}{R}\right)\cos\theta + \frac{1}{4}\left(\frac{r}{R}\right)^2 (3\cos^2\theta - 1) + \dots\right], \end{aligned}}$$

(for $r < R$, northern hemisphere).

In the southern hemisphere we'll have to go for $\theta = \pi$, using $P_l(-1) = (-1)^l$.

$$V(r,\pi) = \sum_{l=0}^{\infty} (-1)^l \overline{A}_l r^l = \frac{\sigma}{2\epsilon_0}\left[\sqrt{r^2+R^2} - r\right].$$

(I put an overbar on $\overline{A}_l$ to distinguish it from the northern A_l). The only difference is the sign of $\overline{A}_1$: $\overline{A}_1 = +(\sigma/2\epsilon_0)$, $\overline{A}_0 = A_0$, $\overline{A}_2 = A_2$. So:

$$\boxed{\begin{aligned} V(r,\theta) &= \frac{\sigma}{2\epsilon_0}\left[R + rP_1(\cos\theta) + \frac{1}{2R}r^2P_2(\cos\theta) + \dots\right], \\ &= \frac{\sigma R}{2\epsilon_0}\left[1 + \left(\frac{r}{R}\right)\cos\theta + \frac{1}{4}\left(\frac{r}{R}\right)^2(3\cos^2\theta - 1) + \dots\right], \end{aligned}} \quad \text{(for } r < R, \text{ southern hemisphere).}$$

Problem 3.22

$$V(r,\theta) = \left\{\begin{array}{ll} \displaystyle\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta), & (r \le R) \text{ (Eq. 3.78)}, \\ \displaystyle\sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta), & (r \ge R) \text{ (Eq. 3.79)}, \end{array}\right\}$$

where $B_l = A_l R^{2l+1}$ (Eq. 3.81) and

$$\begin{aligned} A_l &= \frac{1}{2\epsilon_0 R^{l-1}} \int_0^{\pi} \sigma_0(\theta) P_l(\cos\theta)\sin\theta\, d\theta \quad \text{(Eq. 3.84)} \\ &= \frac{1}{2\epsilon_0 R^{l-1}}\sigma_0 \left\{ \int_0^{\pi/2} P_l(\cos\theta)\sin\theta\, d\theta - \int_{\pi/2}^{\pi} P_l(\cos\theta)\sin\theta\, d\theta \right\} \quad \text{(let } x = \cos\theta) \\ &= \frac{\sigma_0}{2\epsilon_0 R^{l-1}} \left\{ \int_0^1 P_l(x)\, dx - \int_{-1}^0 P_l(x)\, dx \right\}. \end{aligned}$$

Now $P_l(-x) = (-1)^l P_l(x)$, since $P_l(x)$ is even, for even l, and odd, for odd l. Therefore

$$\int_{-1}^0 P_l(x)\, dx = \int_1^0 P_l(-x)\, d(-x) = (-1)^l \int_0^1 P_l(x)\, dx,$$

and hence

$$A_l = \frac{\sigma_0}{2\epsilon_0 R^{l-1}}\left[1 - (-1)^l\right]\int_0^1 P_l(x)\, dx = \left\{\begin{array}{ll} 0, & \text{if } l \text{ is even} \\ \displaystyle\frac{\sigma_0}{\epsilon_0 R^{l-1}}\int_0^1 P_l(x)\, dx, & \text{if } l \text{ is odd} \end{array}\right\}.$$

So $A_0 = A_2 = A_4 = A_6 = 0$, and all we need are A_1, A_3, and A_5.

$$\begin{aligned}
\int_0^1 P_1(x)\,dx &= \int_0^1 x\,dx = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2}.\\
\int_0^1 P_3(x)\,dx &= \frac{1}{2}\int_0^1 (5x^3-3x)\,dx = \frac{1}{2}\left.\left(5\frac{x^4}{4}-3\frac{x^2}{2}\right)\right|_0^1 = \frac{1}{2}\left(\frac{5}{4}-\frac{3}{2}\right) = -\frac{1}{8}.\\
\int_0^1 P_5(x)\,dx &= \frac{1}{8}\int_0^1 (63x^5-70x^3+15x)\,dx = \frac{1}{8}\left.\left(63\frac{x^6}{6}-70\frac{x^4}{4}+15\frac{x^2}{2}\right)\right|_0^1\\
&= \frac{1}{8}\left(\frac{21}{2}-\frac{35}{2}+\frac{15}{2}\right) = \frac{1}{16}(36-35) = \frac{1}{16}.
\end{aligned}$$

Therefore

$$\boxed{A_1 = \frac{\sigma_0}{\epsilon_0}\left(\frac{1}{2}\right);\ A_3 = \frac{\sigma_0}{\epsilon_0 R^2}\left(-\frac{1}{8}\right);\ A_5 = \frac{\sigma_0}{\epsilon_0 R^4}\left(\frac{1}{16}\right);\ \text{etc.}}$$

and

$$\boxed{B_1 = \frac{\sigma_0}{\epsilon_0}R^3\left(\frac{1}{2}\right);\ B_3 = \frac{\sigma_0}{\epsilon_0}R^5\left(-\frac{1}{8}\right);\ B_5 = \frac{\sigma_0}{\epsilon_0}R^7\left(\frac{1}{16}\right);\ \text{etc.}}$$

Thus

$$\boxed{V(r,\theta) = \left\{\begin{array}{ll}
\dfrac{\sigma_0 r}{2\epsilon_0}\left[P_1(\cos\theta) - \dfrac{1}{4}\left(\dfrac{r}{R}\right)^2 P_3(\cos\theta) + \dfrac{1}{8}\left(\dfrac{r}{R}\right)^4 P_5(\cos\theta) + \dots\right], & (r \le R),\\
\dfrac{\sigma_0 R^3}{2\epsilon_0 r^2}\left[P_1(\cos\theta) - \dfrac{1}{4}\left(\dfrac{R}{r}\right)^2 P_3(\cos\theta) + \dfrac{1}{8}\left(\dfrac{R}{r}\right)^4 P_5(\cos\theta) + \dots\right], & (r \ge R).
\end{array}\right\}}$$

Problem 3.23

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} = 0.$$

Look for solutions of the form $V(s,\phi) = S(s)\Phi(\phi)$:

$$\frac{1}{s}\Phi\frac{d}{ds}\left(s\frac{dS}{ds}\right) + \frac{1}{s^2}S\frac{d^2\Phi}{d\phi^2} = 0.$$

Multiply by s^2 and divide by $V = S\Phi$:

$$\frac{s}{S}\Phi\frac{d}{ds}\left(s\frac{dS}{ds}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0.$$

Since the first term involves s only, and the second ϕ only, each is a constant:

$$\frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right) = C_1, \quad \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = C_2, \quad \text{with } C_1 + C_2 = 0.$$

Now C_2 must be negative (else we get exponentials for Φ, which do not return to their original value—as geometrically they *must*— when ϕ is increased by 2π).

$$C_2 = -k^2. \quad \text{Then } \frac{d^2\Phi}{d\phi^2} = -k^2\Phi \Rightarrow \Phi = A\cos k\phi + B\sin k\phi.$$

Moreover, since $\Phi(\phi + 2\pi) = \Phi(\phi)$, k *must be an integer:* $k = 0, 1, 2, 3, \ldots$ (negative integers are just repeats, but $k = 0$ must be included, since $\Phi = A$ (a constant) is OK).

$s\dfrac{d}{ds}\left(s\dfrac{dS}{ds}\right) = k^2 S$ can be solved by $S = s^n$, provided n is chosen right:

$$s\frac{d}{ds}\left(sns^{n-1}\right) = ns\frac{d}{ds}\left(s^n\right) = n^2 ss^{n-1} = n^2 s^n = k^2 S \Rightarrow n = \pm k.$$

Evidently the general solution is $S(s) = Cs^k + Ds^{-k}$, *unless* $k = 0$, in which case we have only *one* solution to a *second-order* equation—namely, $S =$ constant. So we must treat $k = 0$ separately. *One* solution is a constant—but what's the other? Go back to the differential equation for S, and put in $k = 0$:

$$s\frac{d}{ds}\left(s\frac{dS}{ds}\right) = 0 \Rightarrow s\frac{dS}{ds} = \text{constant} = C \Rightarrow \frac{dS}{ds} = \frac{C}{s} \Rightarrow dS = C\frac{ds}{s} \Rightarrow S = C\ln s + D \text{ (another constant).}$$

So the second solution in this case is $\ln s$. [How about Φ? That *too* reduces to a single solution, $\Phi = A$, in the case $k = 0$. What's the second solution here? Well, putting $k = 0$ into the Φ equation:

$$\frac{d^2\Phi}{d\phi^2} = 0 \Rightarrow \frac{d\Phi}{d\phi} = \text{constant} = B \Rightarrow \Phi = B\phi + A.$$

But a term of the form $B\phi$ is unacceptable, since it does not return to its initial value when ϕ is augmented by 2π.] *Conclusion:* The general solution with cylindrical symmetry is

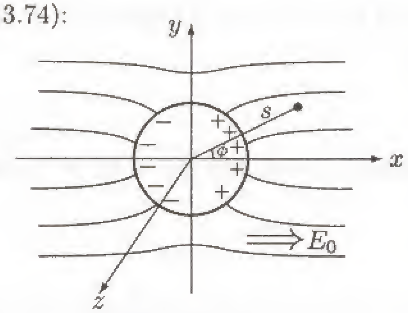
$$\boxed{V(s,\phi) = a_0 + b_0\ln s + \sum_{k=1}^{\infty}\left[s^k(a_k\cos k\phi + b_k\sin k\phi) + s^{-k}(c_k\cos k\phi + d_k\sin k\phi)\right].}$$

Yes: the potential of a line charge goes like $\ln s$, which *is* included.

Problem 3.24

Picking $V = 0$ on the yz plane, with $\mathbf{E}_0$ in the x direction, we have (Eq. 3.74):

$$\left\{\begin{array}{lll} \text{(i)} & V = 0, & \text{when } s = R, \\ \text{(ii)} & V \to -E_0 x = -E_0 s\cos\phi, & \text{for } s \gg R. \end{array}\right\}$$



Evidently $a_0 = b_0 = b_k = d_k = 0$, and $a_k = c_k = 0$ except for $k = 1$:

$$V(s,\phi) = \left(a_1 s + \frac{c_1}{s}\right)\cos\phi.$$

(i)$\Rightarrow$ $c_1 = -a_1 R^2$; (ii)$\rightarrow$ $a_1 = -E_0$. Therefore

$$V(s,\phi) = \left(-E_0 s + \frac{E_0 R^2}{s}\right)\cos\phi, \quad \text{or} \quad \boxed{V(s,\phi) = -E_0 s\left[\left(\frac{R}{s}\right)^2 - 1\right]\cos\phi.}$$

$$\sigma = -\epsilon_0 \left.\frac{\partial V}{\partial s}\right|_{s=R} = -\epsilon_0 E_0 \left(-\frac{R^2}{s^2} - 1\right)\cos\phi\bigg|_{s=R} = \boxed{2\epsilon_0 E_0 \cos\phi.}$$

Problem 3.25

Inside: $V(s,\phi) = a_0 + \sum_{k=1}^{\infty} s^k\left(a_k \cos k\phi + b_k \sin k\phi\right)$. (In this region $\ln s$ and s^{-k} are no good—they blow up at $s = 0$.)

Outside: $V(s,\phi) = \bar{a}_0 + \sum_{k=1}^{\infty} \frac{1}{s^k}\left(c_k \cos k\phi + d_k \sin k\phi\right)$. (Here $\ln s$ and s^k are no good at $s \to \infty$).

$$\sigma = -\epsilon_0 \left(\frac{\partial V_{\text{out}}}{\partial s} - \frac{\partial V_{\text{in}}}{\partial s}\right)\bigg|_{s=R} \quad \text{(Eq. 2.36).}$$

Thus

$$a \sin 5\phi = -\epsilon_0 \sum_{k=1}^{\infty} \left\{-\frac{k}{R^{k+1}}\left(c_k \cos k\phi + d_k \sin k\phi\right) - kR^{k-1}\left(a_k \cos k\phi + b_k \sin k\phi\right)\right\}.$$

Evidently $a_k = c_k = 0$; $b_k = d_k = 0$ except $k = 5$; $a = 5\epsilon_0\left(\frac{1}{R^6}d_5 + R^4 b_5\right)$. Also, V is continuous at $s = R$: $a_0 + R^5 b_5 \sin 5\phi = \bar{a}_0 + \frac{1}{R^5} d_5 \sin 5\phi$. So $a_0 = \bar{a}_0$ (might as well choose both zero); $R^5 b_5 = R^{-5} d_5$, or $d_5 = R^{10} b_5$. Combining these results: $a = 5\epsilon_0\left(R^4 b_5 + R^4 b_5\right) = 10\epsilon_0 R^4 b_5$; $b_5 = \frac{a}{10\epsilon_0 R^4}$; $d_5 = \frac{aR^6}{10\epsilon_0}$. Therefore

$$\boxed{V(s,\phi) = \frac{a \sin 5\phi}{10\epsilon_0}\left\{\begin{array}{ll} s^5/R^4, & \text{for } s < R, \\ R^6/s^5, & \text{for } s > R. \end{array}\right\}}$$

Problem 3.26

Monopole term:

$$Q = \int \rho\, d\tau = kR \int \left[\frac{1}{r^2}(R - 2r)\sin\theta\right] r^2 \sin\theta\, dr\, d\theta\, d\phi.$$

But the r integral is

$$\int_0^R (R - 2r)\, dr = \left(Rr - r^2\right)\Big|_0^R = R^2 - R^2 = 0. \quad \text{So } Q = 0.$$

Dipole term:

$$\int r\cos\theta \rho\, d\tau = kR \int (r\cos\theta)\left[\frac{1}{r^2}(R - 2r)\sin\theta\right] r^2 \sin\theta\, dr\, d\theta\, d\phi.$$

But the θ integral is

$$\int_0^{\pi} \sin^2\theta \cos\theta\, d\theta = \frac{\sin^3\theta}{3}\bigg|_0^{\pi} = \frac{1}{3}(0 - 0) = 0.$$

So the dipole contribution is likewise zero.

Quadrupole term:

$$\int r^2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)\rho\, d\tau = \frac{1}{2}kR \int\!\!\int r^2\left(3\cos^2\theta - 1\right)\left[\frac{1}{r^2}(R - 2r)\sin\theta\right] r^2 \sin\theta\, dr\, d\theta.$$

r integral:

$$\int_0^R r^2(R-2r)\,dr = \left(\frac{r^3}{3}R - \frac{r^4}{2}\right)\Bigg|_0^R = \frac{R^4}{3} - \frac{R^4}{2} = -\frac{R^4}{6}.$$

θ integral:

$$\int_0^\pi \underbrace{(3\cos^2\theta - 1)}_{3(1-\sin^2\theta)-1=2-3\sin^2\theta} \sin^2\theta\,d\theta = 2\int_0^\pi \sin^2\theta\,d\theta - 3\int_0^\pi \sin^4\theta\,d\theta$$

$$= 2\left(\frac{\pi}{2}\right) - 3\left(\frac{3\pi}{8}\right) = \pi\left(1-\frac{9}{8}\right) = -\frac{\pi}{8}.$$

ϕ integral:

$$\int_0^{2\pi} d\phi = 2\pi.$$

The whole integral is:

$$\frac{1}{2}kR\left(-\frac{R^4}{6}\right)\left(-\frac{\pi}{8}\right)(2\pi) = \frac{k\pi^2R^5}{48}.$$

For point P on the z axis ($r \to z$ in Eq. 3.95) the approximate potential is

$$\boxed{V(z) \cong \frac{1}{4\pi\epsilon_0}\frac{k\pi^2R^5}{48z^3}.} \quad \text{(Quadrupole.)}$$

Problem 3.27

$\mathbf{p} = (3qa - qa)\,\hat{\mathbf{z}} + (-2qa - 2q(-a))\,\hat{\mathbf{y}} = 2qa\,\hat{\mathbf{z}}$. Therefore

$$V \cong \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2},$$

and $\mathbf{p}\cdot\hat{\mathbf{r}} = 2qa\,\hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = 2qa\cos\theta$, so

$$V \cong \boxed{\frac{1}{4\pi\epsilon_0}\frac{2qa\cos\theta}{r^2}.} \quad \text{(Dipole.)}$$

Problem 3.28

(a) By symmetry, $\mathbf{p}$ is clearly in the z direction: $\mathbf{p} = p\,\hat{\mathbf{z}}$; $p = \int z\rho\,d\tau \Rightarrow \int z\sigma\,da$.

$$\begin{aligned} p &= \int (R\cos\theta)(k\cos\theta)R^3\sin\theta\,d\theta\,d\phi = 2\pi R^3k\int_0^\pi \cos^2\theta\sin\theta\,d\theta = 2\pi R^3k\left(-\frac{\cos^3\theta}{3}\right)\Bigg|_0^\pi \\ &= \frac{2}{3}\pi R^3k[1-(-1)] = \frac{4\pi R^3k}{3}; \quad \boxed{\mathbf{p} = \frac{4\pi R^3k}{3}\,\hat{\mathbf{z}}.} \end{aligned}$$

(b)

$$V \cong \frac{1}{4\pi\epsilon_0}\frac{4\pi R^3k}{3}\frac{\cos\theta}{r^2} = \boxed{\frac{kR^3}{3\epsilon_0}\frac{\cos\theta}{r^2}.} \quad \text{(Dipole.)}$$

This is *also* the *exact* potential. *Conclusion:* all multiple moments of this distribution (except the dipole) are exactly zero.

Problem 3.29

Using Eq. 3.94 with $r' = d/2$:

$$\frac{1}{\mathscr{r}_+} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{d}{2r}\right)^n P_n(\cos\theta);$$

for $\mathscr{r}_-$, we let $\theta \to 180^\circ + \theta$, so $\cos\theta \to -\cos\theta$:

$$\frac{1}{\mathscr{r}_-} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{d}{2r}\right)^n P_n(-\cos\theta).$$

But $P_n(-x) = (-1)^n P_n(x)$, so

$$V = \frac{1}{4\pi\epsilon_0}q\left(\frac{1}{\mathscr{r}_+} - \frac{1}{\mathscr{r}_-}\right) = \frac{1}{4\pi\epsilon_0}q\frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{d}{2r}\right)^n\left[P_n(\cos\theta) - P_n(-\cos\theta)\right] = \frac{2q}{4\pi\epsilon_0 r}\sum_{n=1,3,5,\dots}\left(\frac{d}{2r}\right)^n P_n(\cos\theta).$$

Therefore

$$V_{\text{dip}} = \frac{2q}{4\pi\epsilon_0}\frac{1}{r}\frac{d}{2r}P_1(\cos\theta) = \frac{qd\cos\theta}{4\pi\epsilon_0 r^2}, \quad \text{while} \quad \boxed{V_{\text{quad}} = 0.}$$

$$V_{\text{oct}} = \frac{2q}{4\pi\epsilon_0 r}\left(\frac{d}{2r}\right)^3 P_3(\cos\theta) = \frac{2q}{4\pi\epsilon_0}\frac{d^3}{8r^4}\frac{1}{2}\left(5\cos^3\theta - 3\cos\theta\right) = \boxed{\frac{qd^3}{4\pi\epsilon_0}\frac{1}{8r^4}\left(5\cos^3\theta - 3\cos\theta\right).}$$

Problem 3.30

(a) (i) $Q = \boxed{2q,}$ (ii) $\mathbf{p} = \boxed{3qa\,\hat{\mathbf{z}},}$ (iii) $V \cong \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}\right] = \boxed{\frac{1}{4\pi\epsilon_0}\left[\frac{2q}{r} + \frac{3qa\cos\theta}{r^2}\right].}$

(b) (i) $Q = \boxed{2q,}$ (ii) $\mathbf{p} = \boxed{qa\,\hat{\mathbf{z}},}$ (iii) $V \cong \boxed{\frac{1}{4\pi\epsilon_0}\left[\frac{2q}{r} + \frac{qa\cos\theta}{r^2}\right].}$

(c) (i) $Q = \boxed{2q,}$ (ii) $\mathbf{p} = \boxed{3qa\,\hat{\mathbf{y}},}$ (iii) $V \cong \boxed{\frac{1}{4\pi\epsilon_0}\left[\frac{2q}{r} + \frac{3qa\sin\theta\sin\phi}{r^2}\right]}$ (from Eq. 1.64, $\hat{\mathbf{y}}\cdot\hat{\mathbf{r}} = \sin\theta\sin\phi$).

Problem 3.31

(a) This point is at $r = a,\ \theta = \frac{\pi}{2},\ \phi = 0$, so $\mathbf{E} = \frac{p}{4\pi\epsilon_0 a^3}\hat{\boldsymbol{\theta}} = \frac{p}{4\pi\epsilon_0 a^3}(-\hat{\mathbf{z}})$; $\mathbf{F} = q\mathbf{E} = \boxed{-\frac{pq}{4\pi\epsilon_0 a^3}\hat{\mathbf{z}}.}$

(b) Here $r = a,\ \theta = 0$, so $\mathbf{E} = \frac{p}{4\pi\epsilon_0 a^3}(2\hat{\mathbf{r}}) = \frac{2p}{4\pi\epsilon_0 a^3}\hat{\mathbf{z}}.$ $\boxed{\mathbf{F} = \frac{2pq}{4\pi\epsilon_0 a^3}\hat{\mathbf{z}}.}$

(c) $V = q\left[V(0,0,a) - V(a,0,0)\right] = \frac{qp}{4\pi\epsilon_0 a^2}\left[\cos(0) - \cos\left(\frac{\pi}{2}\right)\right] = \boxed{\frac{pq}{4\pi\epsilon_0 a^2}.}$

Problem 3.32

$Q = -q$, so $V_{\text{mono}} = \frac{1}{4\pi\epsilon_0}\frac{-q}{r}$; $\mathbf{p} = qa\,\hat{\mathbf{z}}$, so $V_{\text{dip}} = \frac{1}{4\pi\epsilon_0}\frac{qa\cos\theta}{r^2}$. Therefore

$$\boxed{V(r,\theta) \cong \frac{q}{4\pi\epsilon_0}\left(-\frac{1}{r} + \frac{a\cos\theta}{r^2}\right).} \qquad \boxed{\mathbf{E}(r,\theta) \cong \frac{q}{4\pi\epsilon_0}\left[-\frac{1}{r^2}\hat{\mathbf{r}} + \frac{a}{r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right)\right].}$$

Problem 3.33

$\mathbf{p} = (\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} + (\mathbf{p}\cdot\hat{\boldsymbol{\theta}})\,\hat{\boldsymbol{\theta}} = p\cos\theta\,\hat{\mathbf{r}} - p\sin\theta\,\hat{\boldsymbol{\theta}}$ (Fig. 3.36). So $3(\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{p} = 3p\cos\theta\,\hat{\mathbf{r}} - p\cos\theta\,\hat{\mathbf{r}} + p\sin\theta\,\hat{\boldsymbol{\theta}} = 2p\cos\theta\,\hat{\mathbf{r}} + p\sin\theta\,\hat{\boldsymbol{\theta}}$. So Eq. 3.104 ≡ Eq. 3.103. ✓

Problem 3.34

At height x above the plane, the force on q is given by Eq. 3.12: $F = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4z^2} = m\frac{d^2x}{dt^2}$; $\frac{d^2x}{dt^2} = -A/x^2$, where $A \equiv \frac{q^2}{16\pi\epsilon_0 m}$. Multiply by $v = \frac{dx}{dt}$: $v\frac{dv}{dt} = -\frac{A}{x^2}\frac{dx}{dt} \Rightarrow \frac{d}{dt}\left(\frac{1}{2}v^2\right) = \frac{d}{dt}\left(\frac{A}{x}\right) \Rightarrow \frac{1}{2}v^2 = \frac{A}{x} +$ constant. But $v = 0$ when $x = d$, so constant $= -A/d$, and hence $v^2 = 2A\left(\frac{1}{x} - \frac{1}{d}\right)$; $-\frac{dx}{dt} = \sqrt{2A}\sqrt{\frac{1}{x} - \frac{1}{d}} = \sqrt{\frac{2A}{d}}\sqrt{\frac{d-x}{x}}$.

$$\int_d^0 \frac{\sqrt{x}}{\sqrt{d-x}}\,dx = -\sqrt{\frac{2A}{d}}\int_0^t dt = -\sqrt{\frac{2A}{d}}\,t.$$

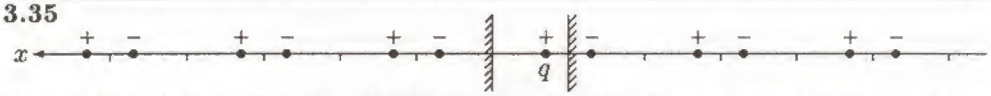
This integral can also be integrated directly. Let $x = u^2$; $dx = 2u\,du$.

$$\int_d^0 \frac{\sqrt{x}}{\sqrt{d-x}}\,dx = 2\int_{\sqrt{d}}^0 \frac{u^2}{\sqrt{d-u^2}}\,du = 2\left\{-\frac{u}{2}\sqrt{d-u^2} + \frac{d}{2}\sin^{-1}\left(\frac{u}{\sqrt{d}}\right)\right\}\Bigg|_{\sqrt{d}}^0 = -d\sin^{-1}(1) = -d\frac{\pi}{2}.$$

Therefore

$$t = \sqrt{\frac{d}{2A}}\frac{\pi d}{2} = \sqrt{\frac{\pi^2 d^2}{4}\frac{d}{2q^2}16\pi\epsilon_0 m} = \boxed{\sqrt{\frac{2\pi^3 d^3 \epsilon_0 m}{q^2}}.}$$

Problem 3.35



The image configuration is shown in the figure; the positive image charge forces cancel in pairs. The net force of the negative image charges is:

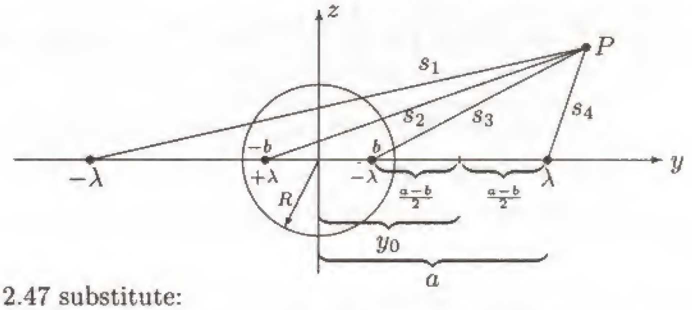
$$\begin{aligned} F &= \frac{1}{4\pi\epsilon_0}q^2\left\{\frac{1}{[2(a-x)]^2} + \frac{1}{[2a+2(a-x)]^2} + \frac{1}{[4a+2(a-x)]^2} + \dots\right. \\ &\qquad \left. -\frac{1}{(2x)^2} - \frac{1}{(2a+2x)^2} - \frac{1}{(4a+2x)^2} - \dots\right\} \\ &= \boxed{\frac{1}{4\pi\epsilon_0}\frac{q^2}{4}\left\{\left[\frac{1}{(a-x)^2} + \frac{1}{(2a-x)^2} + \frac{1}{(3a-x)^2} + \dots\right] - \left[\frac{1}{x^2} + \frac{1}{(a+x)^2} + \frac{1}{(2a+x)^2} + \dots\right]\right\}.} \end{aligned}$$

When $a \to \infty$ (i.e. $a \gg x$) only the $\frac{1}{x^2}$ term survives: $F = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{(2x)^2}$ ✓ (same as for only *one* plane—Eq. 3.12). When $x = a/2$,

$$F = \frac{1}{4\pi\epsilon_0}\frac{q^2}{4}\left\{\left[\frac{1}{(a/2)^2} + \frac{1}{(3a/2)^2} + \frac{1}{(5a/2)^2} + \dots\right] - \left[\frac{1}{(a/2)^2} + \frac{1}{(3a/2)^2} + \frac{1}{(5a/2)^2} + \dots\right]\right\} = 0. \checkmark$$

Problem 3.36

Following Prob. 2.47, we place image line charges $-\lambda$ at $y = b$ and $+\lambda$ at $y = -b$ (here y is the horizontal axis, z vertical).



In the solution to Prob. 2.47 substitute:

$$a \to \frac{a-b}{2},\ y_0 \to \frac{a+b}{2} \text{ so } \left(\frac{a-b}{2}\right)^2 = \left(\frac{a+b}{2}\right)^2 - R^2 \Rightarrow b = \frac{R^2}{a}.$$

$$\begin{aligned} V &= \frac{\lambda}{4\pi\epsilon_0}\left[\ln\left(\frac{s_3^2}{s_4^2}\right) + \ln\left(\frac{s_1^2}{s_2^2}\right)\right] = \frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{s_1^2 s_3^2}{s_4^2 s_2^2}\right) \\ &= \frac{\lambda}{4\pi\epsilon_0}\ln\left\{\frac{[(y+a)^2+z^2][(y-b)^2+z^2]}{[(y-a)^2+z^2][(y+b)^2+z^2]}\right\}, \quad \text{or, using } y = s\cos\phi,\ z = s\sin\phi, \\ &= \boxed{\frac{\lambda}{4\pi\epsilon_0}\ln\left\{\frac{(s^2+a^2+2as\cos\phi)[(as/R)^2+R^2-2as\cos\phi]}{(a^2+a^2-2as\cos\phi)[(as/R)^2+R^2+2as\cos\phi]}\right\}}. \end{aligned}$$

Problem 3.37

Since the configuration is azimuthally symmetric, $V(r,\theta) = \sum\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$.

(a) $r > b$: $A_l = 0$ for all l, since $V \to 0$ at ∞. Therefore $V(r,\theta) = \sum \frac{B_l}{r^{l+1}}P_l(\cos\theta)$.

$a < r < b:\ V(r,\theta) = \sum\left(C_l r^l + \frac{D_l}{r^{l+1}}\right)P_l(\cos\theta)$. $r < a:\ V(r,\theta) = V_0$.

We need to determine B_l, C_l, D_l, and V_0. To do this, invoke boundary conditions as follows: (i) V is continuous at a, (ii) V is continuous at b, (iii) $\Delta\left(\frac{\partial V}{\partial r}\right) = -\frac{1}{\epsilon_0}\sigma(\theta)$ at b.

(ii) $\Rightarrow \sum \frac{B_l}{b^{l+1}}P_l(\cos\theta) = \sum\left(C_l b^l + \frac{D_l}{b^{l+1}}\right)P_l(\cos\theta);\quad \frac{B_l}{b^{l+1}} = C_l b^l + \frac{D_l}{b^{l+1}} \Rightarrow \boxed{B_l = b^{2l+1}C_l + D_l.}$ (1)

(i) $\Rightarrow \sum\left(C_l a^l + \frac{D_l}{a^{l+1}}\right)P_l(\cos\theta) = V_0; \left\{\begin{array}{ll} C_l a^l + \dfrac{D_l}{a^{l+1}} = 0, & \text{if } l \neq 0, \\ C_0 a^0 + \dfrac{D_0}{a^1} = V_0, & \text{if } l = 0; \end{array}\right\} \boxed{\begin{array}{l} D_l = -a^{2l+1}C_l,\ l \neq 0, \\ D_0 = aV_0 - aC_0. \end{array}}$ (2)

Putting (2) into (1) gives $B_l = b^{2l+1}C_l - a^{2l+1}C_l,\ l \neq 0,\quad B_0 = bC_0 + aV_0 - aC_0$. Therefore

$$\boxed{\begin{array}{l} B_l = \left(b^{2l+1} - a^{2l+1}\right)C_l,\ l \neq 0, \\ B_0 = (b-a)C_0 + aV_0. \end{array}} \quad (1')$$

(iii) $\Rightarrow \sum B_l[-(l+1)]\frac{1}{b^{l+2}}P_l(\cos\theta) - \sum\left(C_l l b^{l-1} + D_l\frac{-(l+1)}{b^{l+2}}\right)P_l(\cos\theta) = \frac{-k}{\epsilon_0}P_1(\cos\theta)$. So

$$-\frac{(l+1)}{b^{l+2}}B_l - \left(C_l l b^{l-1} + D_l\frac{-(l+1)}{b^{l+2}}\right) = 0, \text{ if } l \neq 1;$$

or

$$-(l+1)B_l - lC_l b^{2l+1} + (l+1)D_l = 0; \quad (l+1)(B_l - D_l) = -lb^{2l+1}C_l.$$

$$B_1(+2)\frac{1}{b^2} + \left(C_1 + D_1\frac{-2}{b^2}\right) = \frac{k}{\epsilon_0}, \text{ for } l = 1; \quad C_1 + \frac{2}{b^3}(B_1 - D_1) = k.$$

Therefore

$$\boxed{\begin{array}{l} (l+1)(B_l - D_l) + lb^{2l+1}C_l = 0, \text{for } l \neq 1, \\ C_1 + \frac{2}{b^3}(B_1 - D_1) = \frac{k}{\epsilon_0}. \end{array}} \quad (3)$$

Plug (2) and (1′) into (3):

For $l \neq 0$ or 1:

$(l+1)\left[\left(b^{2l+1} - a^{2l+1}\right)C_l + a^{2l+1}C_l\right] + lb^{2l+1}C_l = 0; \quad (l+1)b^{2l+1}C_l + lb^{2l+1}C_l = 0;\ (2l+1)C_l = 0 \Rightarrow C_l = 0.$

Therefore (1′) and (2) $\Rightarrow \boxed{B_l = C_l = D_l = 0 \text{ for } l > 1.}$

For $l = 1$: $C_1 + \frac{2}{b^3}\left[\left(b^3 - a^3\right)C_1 + a^3C_1\right] = k; \quad C_1 + 2C_1 = k \Rightarrow \boxed{C_1 = k/3\epsilon_0;}\ D_1 = -a^3C_1 \Rightarrow \boxed{D_1 = -a^3k/3\epsilon_0;}\quad B_1 = \left(b^3 - a^3\right)C_1 \Rightarrow \boxed{B_1 = \left(b^3 - a^3\right)k/3\epsilon_0.}$

For $l = 0$: $B_0 - D_0 = 0 \Rightarrow B_0 = D_0 \Rightarrow (b-a)C_0 + aV_0 = aV_0 - aC_0$, so $bC_0 = 0 \Rightarrow \boxed{C_0 = 0;\ D_0 = aV_0 = B_0.}$

Conclusion: $\boxed{V(r,\theta) = \frac{aV_0}{r} + \frac{\left(b^3 - a^3\right)k}{3r^2\epsilon_0}\cos\theta,}\ r \geq b.$ $\boxed{V(r,\theta) = \frac{aV_0}{r} + \frac{k}{3\epsilon_0}\left(r - \frac{a^3}{r^2}\right)\cos\theta,}\quad a \leq r \leq b.$

(b)$\sigma_i(\theta) = -\epsilon_0 \left.\frac{\partial V}{\partial r}\right|_a = -\epsilon_0\left[-\frac{aV_0}{a^2} + \frac{k}{3\epsilon_0}\left(1 + 2\frac{a^3}{a^3}\right)\cos\theta\right] = -\epsilon_0\left(-\frac{V_0}{a} + \frac{k}{\epsilon_0}\cos\theta\right) = \boxed{-k\cos\theta + V_0\frac{\epsilon_0}{a}.}$

(c)$q_i = \int \sigma_i\, da = \frac{V_0\epsilon_0}{a}4\pi a^2 = \boxed{4\pi a\epsilon_0 V_0 = Q_{\text{tot}}.}$ At large r: $V \approx \frac{aV_0}{r} \overset{?}{=} \frac{1}{4\pi\epsilon_0}\frac{Q}{r} = \frac{1}{4\pi\epsilon_0}\frac{4\pi a\epsilon_0 V_0}{r} = \frac{aV_0}{r}$. ✓

Problem 3.38

Use multipole expansion (Eq. 3.95): $\rho\, d\tau \rightarrow \lambda\, dz = \frac{Q}{2a}\, dz$, and $r' \rightarrow z$:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int_{-a}^{a} z^n P_n(\cos\theta)\frac{Q}{2a}\, dz.$$

The integral is

$$\frac{Q}{2a}P_n(\cos\theta)\int_{-a}^{a} z^n\, dz = \frac{Q}{2a}P_n(\cos\theta)\left.\frac{z^{n+1}}{n+1}\right|_{-a}^{a} = \frac{Q}{2a}P_n(\cos\theta)\frac{2a^{n+1}}{n+1} \quad \text{for } n \text{ even, zero for } n \text{ odd.}$$

Therefore

$$V = \frac{Q}{4\pi\epsilon_0}\frac{1}{r}\sum_{n=0,2,4,\ldots}\left[\frac{1}{n+1}\left(\frac{a}{r}\right)^n P_n(\cos\theta)\right]. \quad \text{qed}$$

Problem 3.39

Use separation of variables in cylindrical coordinates (Prob. 3.23):

$$V(s,\phi) = a_0 + b_0\ln s + \sum_{k=1}^{\infty}\left[s^k(a_k\cos k\phi + b_k\sin k\phi) + s^{-k}(c_k\cos k\phi + d_k\sin k\phi)\right].$$

$$s<R:\quad V(s,\phi)=\textstyle\sum_{k=1}^{\infty} s^k(a_k\cos k\phi+b_k\sin k\phi)\quad (\ln s \text{ and } s^{-k} \text{ blow up at } s=0);$$
$$s>R:\quad V(s,\phi)=\textstyle\sum_{k=1}^{\infty} s^{-k}(c_k\cos k\phi+d_k\sin k\phi)\quad (\ln s \text{ and } s^{k} \text{ blow up as } s\to\infty).$$

(We may as well pick constants so $V\to 0$ as $s\to\infty$, and hence $a_0=0$.) Continuity at $s=R\Rightarrow$ $\sum R^k(a_k\cos k\phi+b_k\sin k\phi)=\sum R^{-k}(c_k\cos k\phi+d_k\sin k\phi)$, so $c_k=R^{2k}a_k,\ d_k=R^{2k}b_k$. Eq. 2.36 says: $\left.\frac{\partial V}{\partial s}\right|_{R^+}-\left.\frac{\partial V}{\partial s}\right|_{R^-}=-\frac{1}{\epsilon_0}\sigma$. Therefore

$$\sum\frac{-k}{R^{k+1}}(c_k\cos k\phi+d_k\sin k\phi)-\sum kR^{k-1}(a_k\cos k\phi+b_k\sin k\phi)=-\frac{1}{\epsilon_0}\sigma,$$

or:

$$\sum 2kR^{k-1}(a_k\cos k\phi+b_k\sin k\phi)=\left\{\begin{array}{ll}\sigma_0/\epsilon_0 & (0<\phi<\pi)\\ -\sigma_0/\epsilon_0 & (\pi<\phi<2\pi)\end{array}\right\}.$$

Fourier's trick: multiply by $(\cos l\phi)\,d\phi$ and integrate from 0 to 2π, using

$$\int_0^{2\pi}\sin k\phi\cos l\phi\,d\phi=0;\qquad\int_0^{2\pi}\cos\ k\phi\cos l\phi\,d\phi=\left\{\begin{array}{ll}0, & k\neq l\\ \pi, & k=l\end{array}\right\}.$$

Then

$$2lR^{l-1}\pi a_l=\frac{\sigma_0}{\epsilon_0}\left[\int_0^{\pi}\cos l\phi\,d\phi-\int_{\pi}^{2\pi}\cos l\phi\,d\phi\right]=\frac{\sigma_0}{\epsilon_0}\left\{\left.\frac{\sin l\phi}{l}\right|_0^{\pi}-\left.\frac{\sin l\phi}{l}\right|_{\pi}^{2\pi}\right\}=0;\quad a_l=0.$$

Multiply by $(\sin l\phi)\,d\phi$ and integrate, using $\int_0^{2\pi}\sin\ k\phi\sin l\phi\,d\phi=\left\{\begin{array}{ll}0, & k\neq l\\ \pi, & k=l\end{array}\right\}$:

$$\begin{aligned}2lR^{l-1}\pi b_l &= \frac{\sigma_0}{\epsilon_0}\left[\int_0^{\pi}\sin l\phi\,d\phi-\int_{\pi}^{2\pi}\sin l\phi\,d\phi\right]=\frac{\sigma_0}{\epsilon_0}\left\{-\left.\frac{\cos l\phi}{l}\right|_0^{\pi}+\left.\frac{\cos l\phi}{l}\right|_{\pi}^{2\pi}\right\}=\frac{\sigma_0}{l\epsilon_0}(2-2\cos l\pi)\\ &= \left\{\begin{array}{ll}0, & \text{if } l \text{ is even}\\ 4\sigma_0/l\epsilon_0, & \text{if } l \text{ is odd}\end{array}\right\}\Rightarrow b_l=\left\{\begin{array}{ll}0, & \text{if } l \text{ is even}\\ 2\sigma_0/\pi\epsilon_0 l^2R^{l-1}, & \text{if } l \text{ is odd}\end{array}\right\}.\end{aligned}$$

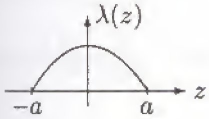
Conclusion:

$$\boxed{V(s,\phi)=\frac{2\sigma_0R}{\pi\epsilon_0}\sum_{k=1,3,5,\dots}\frac{1}{k^2}\sin k\phi\left\{\begin{array}{ll}(s/R)^k & (s<R)\\ (R/s)^k & (s>R)\end{array}\right\}.}$$

Problem 3.40

Use Eq. 3.95, in the form $V(\mathbf{r})=\frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{P_n(\cos\theta)}{r^{n+1}}I_n;\quad I_n=\int_{-a}^{a}z^n\lambda(z)\,dz.$

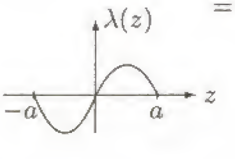
(a) $I_0=k\int_{-a}^{a}\cos\left(\frac{\pi z}{2a}\right)dz=k\left[\frac{2a}{\pi}\sin\left(\frac{\pi z}{2a}\right)\right]\Big|_{-a}^{a}=\frac{2ak}{\pi}\left[\sin\left(\frac{\pi}{2}\right)-\sin\left(-\frac{\pi}{2}\right)\right]=\frac{4ak}{\pi}$. Therefore:



$$\boxed{V(r,\theta)\cong\frac{1}{4\pi\epsilon_0}\left(\frac{4ak}{\pi}\right)\frac{1}{r}.}\quad\text{(Monopole.)}$$

(b)

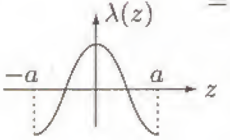
$$\begin{aligned} I_0 &= 0. \\ I_1 &= k\int_{-a}^{a} z\sin(\pi z/a)\,dz = k\left\{\left(\frac{a}{\pi}\right)^2 \sin\left(\frac{\pi z}{a}\right) - \frac{az}{\pi}\cos\left(\frac{\pi z}{a}\right)\right\}\Bigg|_{-a}^{a} \\ &= k\left\{\left(\frac{a}{\pi}\right)^2 [\sin(\pi) - \sin(-\pi)] - \frac{a^2}{\pi}\cos(\pi) - \frac{a^2}{\pi}\cos(-\pi)\right\} = k\frac{2a^2}{\pi}; \end{aligned}$$



$$\boxed{V(r,\theta) \cong \frac{1}{4\pi\epsilon_0}\left(\frac{2a^2k}{\pi}\right)\frac{1}{r^2}\cos\theta.} \quad \text{(Dipole.)}$$

(c)

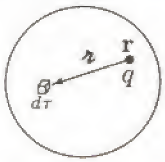
$$\begin{aligned} I_0 &= I_1 = 0. \\ I_2 &= k\int_{-a}^{a} z^2\cos\left(\frac{\pi z}{a}\right)dz = k\left\{\frac{2z\cos(\pi z/a)}{(\pi/a)^2} + \frac{(\pi z/a)^2 - 2}{(\pi/a)^3}\sin\left(\frac{\pi z}{a}\right)\right\}\Bigg|_{-a}^{a} \\ &= 2k\left(\frac{a}{\pi}\right)^2 [a\cos(\pi) + a\cos(-\pi)] = -\frac{4a^3k}{\pi^2}. \end{aligned}$$



$$\boxed{V(r,\theta) \cong \frac{1}{4\pi\epsilon_0}\left(-\frac{4a^3k}{\pi^2}\right)\frac{1}{2r^3}\left(3\cos^2\theta - 1\right).} \quad \text{(Quadrupole.)}$$

Problem 3.41

(a) The average field due to a point charge q at $\mathbf{r}$ is



$$\mathbf{E}_{\text{ave}} = \frac{1}{\left(\frac{4}{3}\pi\epsilon_0 R^3\right)}\int \mathbf{E}\,d\tau, \quad \text{where } \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}},$$

$$\text{so} \quad \mathbf{E}_{\text{ave}} = \frac{1}{\left(\frac{4}{3}\pi\epsilon_0 R^3\right)}\frac{1}{4\pi\epsilon_0}\int \rho\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau.$$

(Here $\mathbf{r}$ is the source point, $d\tau$ is the field point, so $\boldsymbol{\mathscr{r}}$ goes from $\mathbf{r}$ to $d\tau$.) The field at $\mathbf{r}$ due to uniform charge ρ over the sphere is $\mathbf{E}_s = \frac{1}{4\pi\epsilon_0}\int \rho\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau$. This time $d\tau$ is the source point and $\mathbf{r}$ is the field point, so $\boldsymbol{\mathscr{r}}$ goes from $d\tau$ to $\mathbf{r}$, and hence carries the opposite sign. So with $\rho = -q/\left(\frac{4}{3}\pi R^3\right)$, the two expressions agree: $\mathbf{E}_{\text{ave}} = \mathbf{E}_\rho$.

(b) From Prob. 2.12:

$$\mathbf{E}_\rho = \frac{1}{3\epsilon_0}\rho\,\mathbf{r} = -\frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}}{R^3} = -\frac{\mathbf{p}}{4\pi\epsilon_0 R^3}.$$

(c) If there are many charges inside the sphere, $\mathbf{E}_{\text{ave}}$ is the sum of the individual averages, and $\mathbf{p}_{\text{tot}}$ is the sum of the individual dipole moments. So $\mathbf{E}_{\text{ave}} = -\dfrac{\mathbf{p}}{4\pi\epsilon_0 R^3}$. qed

(d) The same argument, only with q placed at $\mathbf{r}$ *outside* the sphere, gives

$$\mathbf{E}_{\text{ave}} = \mathbf{E}_\rho = \frac{1}{4\pi\epsilon_0}\frac{\left(\frac{4}{3}\pi R^3\rho\right)}{r^2}\,\hat{\mathbf{r}} \quad \text{(field at } \mathbf{r} \text{ due to uniformly charged sphere)} = \frac{1}{4\pi\epsilon_0}\frac{-q}{r^2}\,\hat{\mathbf{r}}.$$

But this is precisely the field produced by q (at $\mathbf{r}$) at the *center* of the sphere. So the average field (over the sphere) due to a point charge *outside* the sphere is the same as the field that same charge produces at the center. And by superposition, this holds for any *collection* of exterior charges.

Problem 3.42

(a)

$$\begin{aligned}
\mathbf{E}_{\text{dip}} &= \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}) \\
&= \frac{p}{4\pi\epsilon_0 r^3}\,[2\cos\theta(\sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}) \\
&\quad + \sin\theta(\cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}})] \\
&= \frac{p}{4\pi\epsilon_0 r^3}\left[3\sin\theta\cos\theta\cos\phi\,\hat{\mathbf{x}} + 3\sin\theta\cos\theta\sin\phi\,\hat{\mathbf{y}} + \underbrace{(2\cos^2\theta - \sin^2\theta)}_{=3\cos^2\theta-1}\,\hat{\mathbf{z}}\right]. \\
\mathbf{E}_{\text{ave}} &= \frac{1}{\left(\frac{4}{3}\pi R^3\right)}\int \mathbf{E}_{\text{dip}}\,d\tau \\
&= \frac{1}{\left(\frac{4}{3}\pi R^3\right)}\left(\frac{p}{4\pi\epsilon_0}\right)\int \frac{1}{r^3}\left[3\sin\theta\cos\theta(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}) + (3\cos^2\theta - 1)\;\hat{\mathbf{z}}\right] r^2\sin\theta\,dr\,d\theta\,d\phi.
\end{aligned}$$

But $\int\limits_0^{2\pi}\cos\phi\,d\phi = \int\limits_0^{2\pi}\sin\phi\,d\phi = 0$, so the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ terms drop out, and $\int\limits_0^{2\pi} d\phi = 2\pi$, so

$$\mathbf{E}_{\text{ave}} = \frac{1}{\left(\frac{4}{3}\pi R^3\right)}\left(\frac{p}{4\pi\epsilon_0}\right)2\pi\int\limits_0^R \frac{1}{r}\,dr\;\;\underbrace{\int\limits_0^\pi (3\cos^2\theta - 1)\sin\theta\,d\theta}_{(-\cos^3\theta+\cos\theta)|_0^\pi=1-1+1-1=0}\;.$$

Evidently $\boxed{\mathbf{E}_{\text{ave}} = 0,}$ which contradicts the result of Prob. 3.41. [Note, however, that the r integral, $\int_0^R \frac{1}{r}\,dr$, blows up, since $\ln r \to -\infty$ as $r \to 0$. If, as suggested, we truncate the r integral at $r = \epsilon$, then it is finite, and the θ integral gives $\mathbf{E}_{\text{ave}} = 0$.]

(b) We want $\mathbf{E}$ within the ϵ-sphere to be a delta function: $\mathbf{E} = \mathbf{A}\delta^3(\mathbf{r})$, with $\mathbf{A}$ selected so that the *average* field is consistent with the general theorem in Prob. 3.41:

$$\mathbf{E}_{\text{ave}} = \frac{1}{\left(\frac{4}{3}\pi R^3\right)}\int \mathbf{A}\delta^3(\mathbf{r})\,d\tau = \frac{\mathbf{A}}{\left(\frac{4}{3}\pi R^3\right)} = -\frac{\mathbf{p}}{4\pi\epsilon_0 R^3} \Rightarrow \mathbf{A} = -\frac{\mathbf{p}}{3\epsilon_0},\text{ and hence }\boxed{\mathbf{E} = -\frac{\mathbf{p}}{3\epsilon_0}\delta^3(\mathbf{r}).}$$

Problem 3.43

(a) $I = \int(\boldsymbol{\nabla}V_1)\cdot(\boldsymbol{\nabla}V_2)\,d\tau$. But $\boldsymbol{\nabla}\cdot(V_1\boldsymbol{\nabla}V_2) = (\boldsymbol{\nabla}V_1)\cdot(\boldsymbol{\nabla}V_2) + V_1(\nabla^2V_2)$, so

$$I = \int \boldsymbol{\nabla}\cdot(V_1\boldsymbol{\nabla}V_2)\,d\tau - \int V_1(\nabla^2V_2) = \oint_S V_1(\boldsymbol{\nabla}V_2)\cdot d\mathbf{a} + \frac{1}{\epsilon_0}\int V_1\rho_2\,d\tau.$$

But the surface integral is over a huge sphere "at infinity", where V_1 and $V_2 \to 0$. So $I = \frac{1}{\epsilon_0}\int V_1\rho_2\,d\tau$. By the same argument, with 1 and 2 reversed, $I = \frac{1}{\epsilon_0}\int V_2\rho_1\,d\tau$. So $\int V_1\rho_2\,d\tau = \int V_2\rho_1\,d\tau$. qed

(b) $$\left\{\begin{array}{l} \textit{Situation (1)}: Q_a = \int_a \rho_1\, d\tau = Q;\;\; Q_b = \int_b \rho_1\, d\tau = 0;\;\; V_{1b} \equiv V_{ab}. \\ \textit{Situation (2)}: Q_a = \int_a \rho_2\, d\tau = 0;\;\; Q_b = \int_b \rho_2\, d\tau = Q;\;\; V_{2a} \equiv V_{ba}. \end{array}\right\}$$

$$\left\{\begin{array}{l} \int V_1\rho_2\, d\tau = V_{1a}\int_a \rho_2\, d\tau + V_{1b}\int_b \rho_2\, d\tau = V_{ab}Q. \\ \int V_2\rho_1\, d\tau = V_{2a}\int_a \rho_1\, d\tau + V_{2b}\int_b \rho_1\, d\tau = V_{ba}Q. \end{array}\right\}$$

Green's reciprocity theorem says $QV_{ab} = QV_{ba}$, so $V_{ab} = V_{ba}$. qed

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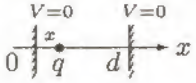
Problem 3.44

(a) *Situation (1):* actual. *Situation (2):* right plate at V_0, left plate at $V = 0$, no charge at x.



$$\int V_1\rho_2\, d\tau = V_{l_1}Q_{l_2} + V_{x_1}Q_{x_2} + V_{r_1}Q_{r_2}.$$

But $V_{l_1} = V_{r_1} = 0$ and $Q_{x_2} = 0$, so $\int V_1\rho_2\, d\tau = 0$.

$$\int V_2\rho_1\, d\tau = V_{l_2}Q_{l_1} + V_{x_2}Q_{x_1} + V_{r_2}Q_{r_1}.$$

But $V_{l_2} = 0$ $Q_{x_1} = q$, $V_{r_2} = V_0$, $Q_{r_1} = Q_2$, and $V_{x_2} = V_0(x/d)$. So $0 = V_0(x/d)q + V_0Q_2$, and hence

$$\boxed{Q_2 = -qx/d.}$$

Situation (1): actual. *Situation (2):* left plate at V_0, right plate at $V = 0$, no charge at x.

$$\int V_1\rho_2\, d\tau = 0 = \int V_2\rho_1\, d\tau = V_{l_2}Q_{l_1} + V_{x_2}Q_{x_1} + V_{r_2}Q_{r_1} = V_0Q_1 + qV_{x_2} + 0.$$

But $V_{x_2} = V_0\left(1 - \frac{x}{d}\right)$, so

$$\boxed{Q_1 = -q(1 - x/d).}$$

(b) *Situation (1):* actual. *Situation (2):* inner sphere at V_0, outer sphere at zero, no charge at r.

$$\int V_1\rho_2\, d\tau = V_{a_1}Q_{a_2} + V_{r_1}Q_{r_2} + V_{b_1}Q_{b_2}.$$

But $V_{a_1} = V_{b_1} = 0$, $Q_{r_2} = 0$. So $\int V_1\rho_2\, d\tau = 0$.

$$\int V_2\rho_1\, d\tau = V_{a_2}Q_{a_1} + V_{r_2}Q_{r_1} + V_{b_2}Q_{b_1} = Q_aV_0 + qV_{r_2} + 0.$$

But V_{r_2} is the potential at r in configuration 2: $V(r) = A + B/r$, with $V(a) = V_0 \Rightarrow A + B/a = V_0$, or $aA + B = aV_0$, and $V(b) = 0 \Rightarrow A + B/b = 0$, or $bA + B = 0$. Subtract: $(b-a)A = -aV_0 \Rightarrow A = -aV_0/(b-a)$; $B\left(\frac{1}{a} - \frac{1}{b}\right) = V_0 = B\frac{(b-a)}{ab} \Rightarrow B = abV_0/(b-a)$. So $V(r) = \frac{aV_0}{(b-a)}\left(\frac{b}{r} - 1\right)$. Therefore

$$Q_aV_0 + q\frac{aV_0}{(b-a)}\left(\frac{b}{r} - 1\right) = 0; \quad \boxed{Q_a = -\frac{qa}{(b-a)}\left(\frac{b}{r} - 1\right).}$$

Now let *Situation (2)* be: inner sphere at zero, outer at V_0, no charge at r.

$$\int V_1\rho_2\,d\tau = 0 = \int V_2\rho_1\,d\tau = V_{a_2}Q_{a_1} + V_{r_2}Q_{r_1} + V_{b_2}Q_{b_1} = 0 + qV_{r_2} + Q_bV_0.$$

This time $isplaystyle V(r) = A + \frac{B}{r}$ with $V(a) = 0 \Rightarrow A + B/a = 0$; $V(b) = V_0 \Rightarrow A + B/b = V_0$, so $V(r) = \frac{bV_0}{(b-a)}\left(1 - \frac{a}{r}\right)$. Therefore, $q\frac{bV_0}{(b-a)}\left(1 - \frac{a}{r}\right) + Q_bV_0 = 0$; $\boxed{Q_b = -\frac{qb}{(b-a)}\left(1 - \frac{a}{r}\right).}$

Problem 3.45

(a) $$\frac{1}{2}\sum_{i,j=1}^{3}\hat{\mathbf{r}}_i\hat{\mathbf{r}}_jQ_{ij} = \frac{1}{2}\int\left\{3\sum_{i=1}^{3}\hat{\mathbf{r}}_ir'_i\sum_{j=1}^{3}\hat{\mathbf{r}}_jr'_j - (r')^2\sum_{i,j}\hat{\mathbf{r}}_i\hat{\mathbf{r}}_j\delta_{ij}\right\}\rho\,d\tau'$$

But $\displaystyle\sum_{i=1}^{3}\hat{\mathbf{r}}_ir'_i = \hat{\mathbf{r}}\cdot\mathbf{r}' = r'\cos\theta' = \sum_{j=1}^{3}\hat{\mathbf{r}}_jr'_j$; $\displaystyle\sum_{i,j}\hat{\mathbf{r}}_i\hat{\mathbf{r}}_j\delta_{ij} = \sum\hat{\mathbf{r}}_j\hat{\mathbf{r}}_j = \hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = 1$. So

$V_{\text{quad}} = \dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r^3}\displaystyle\int\frac{1}{2}\left(r'^2\cos^2\theta' - r'^2\right)\rho\,d\tau' = \dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r^3}\displaystyle\int r'^2P_2(\cos\theta')\rho\,d\tau'$ (the $n = 2$ term in Eq. 3.95).

(b) Because $x^2 = y^2 = (a/2)^2$ for all four charges, $Q_{xx} = Q_{yy} = \left[3(a/2)^2 - (\sqrt{2}a/2)^2\right](q - q - q + q) = 0$. Because $z = 0$ for all four charges, $Q_{zz} = -(\sqrt{2}a/2)^2(q - q - q + q) = 0$ and $Q_{xz} = Q_{yz} = Q_{zx} = Q_{zy} = 0$. This leaves only

$$Q_{xy} = Q_{yx} = 3\left[\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)q + \left(\frac{a}{2}\right)\left(-\frac{a}{2}\right)(-q) + \left(-\frac{a}{2}\right)\left(\frac{a}{2}\right)(-q) + \left(-\frac{a}{2}\right)\left(-\frac{a}{2}\right)q\right] = \boxed{3a^2q.}$$

(c)

$$\begin{aligned}\overline{Q}_{ij} &= \int\left[3(r_i - d_i)(r_j - d_j) - (\mathbf{r} - \mathbf{d})^2\delta_{ij}\right]\rho\,d\tau \quad \text{(I'll drop the primes, for simplicity.)}\\ &= \int\left[3r_ir_j - r^2\delta_{ij}\right]\rho\,d\tau - 3d_i\int r_j\rho\,d\tau - 3d_j\int r_i\rho\,d\tau + 3d_id_j\int\rho\,d\tau + 2\mathbf{d}\cdot\int\mathbf{r}\rho\,d\tau\,\delta_{ij}\\ &\quad - d^2\delta_{ij}\int\rho\,d\tau = Q_{ij} - 3(d_ip_j + d_jp_i) + 3d_id_jQ + 2\delta_{ij}\mathbf{d}\cdot\mathbf{p} - d^2\delta_{ij}Q.\end{aligned}$$

So if $\mathbf{p} = 0$ and $Q = 0$ then $\overline{Q}_{ij} = Q_{ij}$. qed

(d) Eq. 3.95 with $n = 3$:

$$V_{\text{oct}} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^4}\int(r')^3P_3(\cos\theta')\rho\,d\tau'; \quad P_3(\cos\theta) = \frac{1}{2}\left(5\cos^3\theta - 3\cos\theta\right).$$

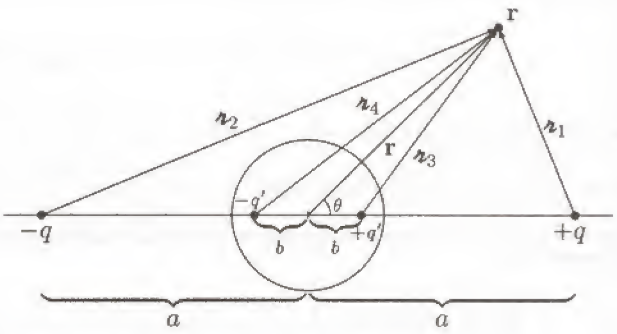
$$\boxed{V_{\text{oct}} = \frac{1}{4\pi\epsilon_0}\frac{\left(\frac{1}{2}\sum_{i,j,k}\hat{\mathbf{r}}_i\hat{\mathbf{r}}_j\hat{\mathbf{r}}_kQ_{ijk}\right)}{r^4},}$$

Define the "octopole moment" as

$$\boxed{Q_{ijk} \equiv \int\left(5r'_ir'_jr'_k - (r')^2(r'_i\delta_{jk} + r'_j\delta_{ik} + r'_k\delta_{ij}\right)\rho(\mathbf{r}')\,d\tau'.}$$

Problem 3.46

$$V = \frac{1}{4\pi\epsilon_0}\left\{q\left(\frac{1}{\mathscr{r}_1} - \frac{1}{\mathscr{r}_2}\right) + q'\left(\frac{1}{\mathscr{r}_3} - \frac{1}{\mathscr{r}_4}\right)\right\}$$



$$\begin{aligned}
\mathscr{r}_1 &= \sqrt{r^2 + a^2 - 2ra\cos\theta},\\
\mathscr{r}_2 &= \sqrt{r^2 + a^2 + 2ra\cos\theta},\\
\mathscr{r}_3 &= \sqrt{r^2 + b^2 - 2rb\cos\theta},\\
\mathscr{r}_4 &= \sqrt{r^2 + b^2 + 2rb\cos\theta}.
\end{aligned}$$

Expanding as in Ex. 3.10: $\left(\frac{1}{\mathscr{r}_1} - \frac{1}{\mathscr{r}_2}\right) \cong \frac{2r}{a^2}\cos\theta$ (we want $a \gg r$, not $r \gg a$, this time).

$$\begin{aligned}
\left(\frac{1}{\mathscr{r}_3} - \frac{1}{\mathscr{r}_4}\right) &\cong \frac{2b}{r^2}\cos\theta \text{ (here we want } b \ll r, \text{ because } b = R^2/a, \text{ Eq. 3.16)}\\
&= \frac{2}{a}\frac{R^2}{r^2}\cos\theta.
\end{aligned}$$

But $q' = -\frac{R}{a}q$ (Eq. 3.15), so

$$V(r,\theta) \cong \frac{1}{4\pi\epsilon_0}\left[q\frac{2r}{a^2}\cos\theta - \frac{R}{a}q\frac{2}{a}\frac{R^2}{r^2}\cos\theta\right] = \frac{1}{4\pi\epsilon_0}\left(\frac{2q}{a^2}\right)\left(r - \frac{R^3}{r^2}\right)\cos\theta.$$

Set $E_0 = -\frac{1}{4\pi\epsilon_0}\frac{2q}{a^2}$ (field in the vicinity of the sphere produced by $\pm q$):

$$\boxed{V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta} \quad \text{(agrees with Eq. 3.76).}$$

Problem 3.47

The boundary conditions are

$$\left.\begin{array}{ll}
\text{(i)} & V = 0 \text{ when } y = 0,\\
\text{(ii)} & V = V_0 \text{ when } y = a,\\
\text{(iii)} & V = 0 \text{ when } x = b,\\
\text{(iv)} & V = 0 \text{ when } x = -b.
\end{array}\right\}$$

Go back to Eq. 3.26 and examine the case $k = 0$: $d^2X/dx^2 = d^2Y/dy^2 = 0$, so $X(x) = Ax + B$, $Y(y) = Cy + D$. But this configuration is symmetric in x, so $A = 0$, and hence the $k = 0$ solution is $V(x,y) = Cy + D$. Pick $D = 0$, $C = V_0/a$, and subtract off this part:

$$V(x,y) = V_0\frac{y}{a} + \bar{V}(x,y).$$

The *remainder* ($\bar{V}(x,y)$) satisfies boundary conditions similar to Ex. 3.4:

$$\left.\begin{array}{ll}
\text{(i)} & \bar{V} = 0 \text{ when } y = 0,\\
\text{(ii)} & \bar{V} = 0 \text{ when } y = a,\\
\text{(iii)} & \bar{V} = -V_0(y/a) \text{ when } x = b,\\
\text{(iv)} & \bar{V} = -V_0(y/a) \text{ when } x = -b.
\end{array}\right\}$$

(The point of peeling off $V_0(y/a)$ was to recover (ii), on which the constraint $k = n\pi/a$ depends.)

The solution (following Ex. 3.4) is

$$\bar{V}(x,y) = \sum_{n=1}^{\infty} C_n \cosh(n\pi x/a)\sin(n\pi y/a),$$

and it remains to fit condition (iii):

$$\bar{V}(b,y) = \sum C_n \cosh(n\pi b/a)\sin(n\pi y/a) = -V_0(y/a).$$

Invoke Fourier's trick:

$$\sum C_n \cosh(n\pi b/a)\int_0^a \sin(n\pi y/a)\sin(n'\pi y/a)\,dy = -\frac{V_0}{a}\int_0^a y\sin(n'\pi y/a)\,dy,$$

$$\frac{a}{2}C_n \cosh(n\pi b/a) = -\frac{V_0}{a}\int_0^a y\sin(n\pi y/a)\,dy.$$

$$\begin{aligned} C_n &= -\frac{2V_0}{a^2\cosh(n\pi b/a)}\left[\left(\frac{a}{n\pi}\right)^2 \sin(n\pi y/a) - \left(\frac{ay}{n\pi}\right)\cos(n\pi y/a)\right]\Bigg|_0^a \\ &= \frac{2V_0}{a^2\cosh(n\pi b/a)}\left(\frac{a^2}{n\pi}\right)\cos(n\pi) = \frac{2V_0}{n\pi}\frac{(-1)^n}{\cosh(n\pi b/a)}. \end{aligned}$$

$$V(x,y) = \boxed{V_0\left[\frac{y}{a} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{\cosh(n\pi x/a)}{\cosh(n\pi b/a)}\sin(n\pi y/a)\right]}.$$

Problem 3.48

(a) Using Prob. 3.14b (with $b = a$):

$$V(x,y) = \frac{4V_0}{\pi}\sum_{n\ \text{odd}}\frac{\sinh(n\pi x/a)\sin(n\pi y/a)}{n\sinh(n\pi)}.$$

$$\begin{aligned} \sigma(y) &= -\epsilon_0\frac{\partial V}{\partial x}\bigg|_{x=0} = -\epsilon_0\frac{4V_0}{\pi}\sum_{n\ \text{odd}}\left(\frac{n\pi}{a}\right)\frac{\cosh(n\pi x/a)\sin(n\pi y/a)}{n\sinh(n\pi)}\bigg|_{x=0} \\ &= -\frac{4\epsilon_0 V_0}{a}\sum_{n\ \text{odd}}\frac{\sin(n\pi y/a)}{\sinh(n\pi)}. \\ \lambda &= \int_0^a \sigma(y)\,dy = -\frac{4\epsilon_0 V_0}{a}\sum_{n\ \text{odd}}\frac{1}{\sinh(n\pi)}\int_0^a \sin(n\pi y/a)\,dy. \\ &\quad \text{But } \int_0^a \sin(n\pi y/a)\,dy = -\frac{a}{n\pi}\cos(n\pi y/a)\big|_0^a = \frac{a}{n\pi}[1-\cos(n\pi)] = \frac{2a}{n\pi}\text{(since } n \text{ is odd).} \\ &= -\frac{8\epsilon_0 V_0}{\pi}\sum_{n\ \text{odd}}\frac{1}{n\sinh(n\pi)} = \boxed{-\frac{\epsilon_0 V_0}{\pi}\ln 2.} \end{aligned}$$

[I have not found a way to sum this series analytically. Mathematica gives the numerical value 0.0866434, which agrees precisely with ln 2/8.]

Using Prob. 3.47 (with $b = a/2$):

$$V(x,y) = V_0\left[\frac{y}{a} + \frac{2}{\pi}\sum_n \frac{(-1)^n\cosh(n\pi x/a)\sin(n\pi y/a)}{n\cosh(n\pi/2)}\right].$$

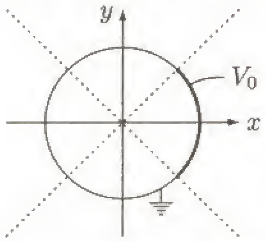
$$\begin{aligned}
\sigma(x) &= -\epsilon_0\frac{\partial V}{\partial y}\Big|_{y=0} = -\epsilon_0 V_0\left[\frac{1}{a} + \frac{2}{\pi}\sum_n\left(\frac{n\pi}{a}\right)\frac{(-1)^n\cosh(n\pi x/a)\cos(n\pi y/a)}{n\cosh(n\pi/2)}\right]\Big|_{y=0} \\
&= -\epsilon_0 V_0\left[\frac{1}{a} + \frac{2}{a}\sum_n \frac{(-1)^n\cosh(n\pi x/a)}{\cosh(n\pi/2)}\right] = -\frac{\epsilon_0 V_0}{a}\left[1 + 2\sum_n \frac{(-1)^n\cosh(n\pi x/a)}{\cosh(n\pi/2)}\right]. \\
\lambda &= \int_{-a/2}^{a/2}\sigma(x)\,dx = -\frac{\epsilon_0 V_0}{a}\left[a + 2\sum_n \frac{(-1)^n}{\cosh(n\pi/2)}\int_{-a/2}^{a/2}\cosh(n\pi x/a)\,dx\right]. \\
&\quad \text{But } \int_{-a/2}^{a/2}\cosh(n\pi x/a)\,dx = \frac{a}{n\pi}\sinh(n\pi x/a)\Big|_{-a/2}^{a/2} = \frac{2a}{n\pi}\sinh(n\pi/2). \\
&= -\frac{\epsilon_0 V_0}{a}\left[a + \frac{4a}{\pi}\sum_n \frac{(-1)^n\tanh(n\pi/2)}{n}\right] = -\epsilon_0 V_0\left[1 + \frac{4}{\pi}\sum_n \frac{(-1)^n\tanh(n\pi/2)}{n}\right] \\
&= \boxed{-\frac{\epsilon_0 V_0}{\pi}\ln 2.}
\end{aligned}$$

[Again, I have not found a way to sum this series analytically. The numerical value is -0.612111, which agrees with the expected value $(\ln 2 - \pi)/4$.]

(b) From Prob. 3.23:

$$V(s,\phi) = a_0 + b_0\ln s + \sum_{k=1}^{\infty}\left(a_k s^k + b_k\frac{1}{s^k}\right)[c_k\cos(k\phi) + d_k\sin(k\phi)].$$

In the interior ($s < R$) b_0 and b_k must be zero ($\ln s$ and $1/s$ blow up at the origin). Symmetry $\Rightarrow d_k = 0$. So



$$V(s,\phi) = a_0 + \sum_{k=1}^{\infty} a_k s^k\cos(k\phi).$$

At the surface:

$$V(R,\phi) = \sum_{k=0} a_k R^k\cos(k\phi) = \begin{cases} V_0, & \text{if } -\pi/4 < \phi < \pi/4, \\ 0, & \text{otherwise.}\end{cases}$$

Fourier's trick: multiply by $\cos(k'\phi)$ and integrate from $-\pi$ to π:

$$\sum_{k=0}^{\infty} a_k R^k\int_{-\pi}^{\pi}\cos(k\phi)\cos(k'\phi)\,d\phi = V_0\int_{-\pi/4}^{\pi/4}\cos(k'\phi)\,d\phi = \begin{cases} V_0\sin(k'\phi)/k'\Big|_{-\pi/4}^{\pi/4} = (V_0/k')\sin(k'\pi/4), & \text{if } k' \neq 0, \\ V_0\pi/2, \text{ if } k' = 0. & \end{cases}$$

But

$$\int_{-\pi}^{\pi}\cos(k\phi)\cos(k'\phi)\,d\phi = \begin{cases} 0, & \text{if } k \neq k' \\ 2\pi, & \text{if } k = k' = 0, \\ \pi, & \text{if } k = k' \neq 0.\end{cases}$$

So $2\pi a_0 = V_0\pi/2 \Rightarrow a_0 = V_0/4$; $\pi a_k R^k = (2V_0/k)\sin(k\pi/4) \Rightarrow a_k = (2V_0/\pi k R^k)\sin(k\pi/4)$ $(k \neq 0)$; hence

$$V(s,\phi) = V_0\left[\frac{1}{4} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\sin(k\pi/4)}{k}\left(\frac{s}{R}\right)^k\cos(k\phi)\right].$$

Using Eq. 2.49, and noting that in this case $\hat{\mathbf{n}} = -\hat{\mathbf{s}}$:

$$\sigma(\phi) = \epsilon_0\frac{\partial V}{\partial s}\bigg|_{s=R} = \epsilon_0 V_0\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\sin(k\pi/4)}{kR^k}ks^{k-1}\cos(k\phi)\bigg|_{s=R} = \frac{2\epsilon_0 V_0}{\pi R}\sum_{k=1}^{\infty}\sin(k\pi/4)\cos(k\phi).$$

We want the net (line) charge on the segment opposite to V_0 $(-\pi < \phi < -3\pi/4$ and $3\pi/4 < \phi < \pi)$:

$$\begin{aligned}\lambda &= \int\sigma(\phi)R\,d\phi = 2R\int_{3\pi/4}^{\pi}\sigma(\phi)\,d\phi = \frac{4\epsilon_0 V_0}{\pi}\sum_{k=1}^{\infty}\sin(k\pi/4)\int_{3\pi/4}^{\pi}\cos(k\phi)\,d\phi \\ &= \frac{4\epsilon_0 V_0}{\pi}\sum_{k=1}^{\infty}\sin(k\pi/4)\left[\frac{\sin(k\phi)}{k}\bigg|_{3\pi/4}^{\pi}\right] = -\frac{4\epsilon_0 V_0}{\pi}\sum_{k=1}^{\infty}\frac{\sin(k\pi/4)\sin(3k\pi/4)}{k}.\end{aligned}$$

k	$\sin(k\pi/4)$	$\sin(3k\pi/4)$	product
1	$1/\sqrt{2}$	$1/\sqrt{2}$	1/2
2	1	-1	-1
3	$1/\sqrt{2}$	$1/\sqrt{2}$	1/2
4	0	0	0
5	$-1/\sqrt{2}$	$-1/\sqrt{2}$	1/2
6	-1	1	-1
7	$-1/\sqrt{2}$	$-1/\sqrt{2}$	1/2
8	0	0	0

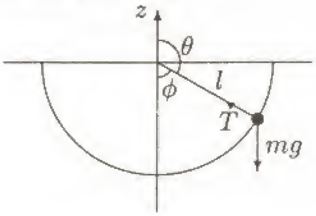
$$\lambda = -\frac{4\epsilon_0 V_0}{\pi}\left[\frac{1}{2}\sum_{1,3,5\ldots}\frac{1}{k} - \sum_{2,6,10,\ldots}\frac{1}{k}\right] = -\frac{4\epsilon_0 V_0}{\pi}\left[\frac{1}{2}\sum_{1,3,5\ldots}\frac{1}{k} - \frac{1}{2}\sum_{1,3,5,\ldots}\frac{1}{k}\right] = 0.$$

Ouch! What went wrong? The problem is that the series $\sum(1/k)$ is divergent, so the "subtraction" $\infty - \infty$ is suspect. One way to avoid this is to go back to $V(s,\phi)$, calculate $\epsilon_0(\partial V/\partial s)$ at $s \neq R$, and save the limit

$s \to R$ until the end:

$$\begin{aligned}
\sigma(\phi, s) &\equiv \epsilon_0 \frac{\partial V}{\partial s} = \frac{2\epsilon_0 V_0}{\pi} \sum_{k=1}^{\infty} \frac{\sin(k\pi/4)}{k} \frac{k s^{k-1}}{R^k} \cos(k\phi) \\
&= \frac{2\epsilon_0 V_0}{\pi R} \sum_{k=1}^{\infty} x^{k-1} \sin(k\pi/4) \cos(k\phi) \quad \text{(where } x \equiv s/R \to 1 \text{ at the end).} \\
\lambda(x) &\equiv \sigma(\phi, s) R\, d\phi = -\frac{4\epsilon_0 V_0}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} x^{k-1} \sin(k\pi/4) \sin(3k\pi/4) \\
&= -\frac{4\epsilon_0 V_0}{\pi} \left[\frac{1}{2x} \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots \right) - \frac{1}{x} \left(\frac{x^2}{2} + \frac{x^6}{6} + \frac{x^{10}}{10} + \cdots \right) \right] \\
&= -\frac{2\epsilon_0 V_0}{\pi x} \left[\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots \right) - \left(x^2 + \frac{x^6}{3} + \frac{x^{10}}{5} + \cdots \right) \right]. \\
& \text{But (see math tables)} : \ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots \right). \\
&= -\frac{2\epsilon_0 V_0}{\pi x} \left[\frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) - \frac{1}{2} \ln\left(\frac{1+x^2}{1-x^2}\right) \right] = -\frac{\epsilon_0 V_0}{\pi x} \ln\left[\left(\frac{1+x}{1-x}\right) \left(\frac{1+x^2}{1-x^2}\right) \right] \\
&= -\frac{\epsilon_0 V_0}{\pi x} \ln\left[\frac{(1+x)^2}{1+x^2} \right]; \quad \lambda = \lim_{x\to 1} \lambda(x) = \boxed{\frac{-\epsilon_0 V_0}{\pi} \ln 2.}
\end{aligned}$$

Problem 3.49



$$\mathbf{F} = q\mathbf{E} = \frac{qp}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}).$$

Now consider the pendulum: $\mathbf{F} = -mg\,\hat{\mathbf{z}} - T\,\hat{\mathbf{r}}$, where $T - mg\cos\phi = mv^2/l$ and (by conservation of energy) $mgl\cos\phi = (1/2)mv^2 \Rightarrow v^2 = 2gl\cos\phi$ (assuming it started from rest at $\phi = 90°$, as stipulated). But $\cos\phi = -\cos\theta$, so $T = mg(-\cos\theta) + (m/l)(-2gl\cos\theta) = -3mg\cos\theta$, and hence

$$\mathbf{F} = -mg(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}) + 3mg\cos\theta\,\hat{\mathbf{r}} = mg(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}).$$

This total force is such as to keep the pendulum on a circular arc, and it is identical to the force on q in the field of a dipole, with $mg \leftrightarrow qp/4\pi\epsilon_0 l^3$. Evidently q also executes semicircular motion, as though it were on a tether of fixed length l.